Singular Control of Optional Random Measures

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Stochastic Optimization and Representation Problems Arising in the Microeconomic Theory of Intertemporal Consumption Choice

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Abstract

In this thesis, we study the problem of maximizing certain concave functionals on the space of optional random measures. Such functionals arise in microeconomic theory where their maximization corresponds to finding the optimal consumption plan of some economic agent.

As an alternative to the well–known methods of Dynamic Programming, we develop a new approach which allows us to clarify the structure of maximizing measures in a general stochastic setting extending beyond the usually required Markovian framework. Our approach is based on an infinite–dimensional version of the Kuhn–Tucker Theorem. The implied first–order conditions allow us to reduce the maximization problem to a new type of representation problem for optional processes which serves as a non–Markovian substitute for the Hamilton–Jacobi–Bellman equation of Dynamic Programming.

In order to solve this representation problem in the deterministic case, we introduce a time–inhomogeneous generalization of convexity. The stochastic case is solved by using an intimate relation to the theory of Gittins–indices in optimal dynamic scheduling. Closed–form solutions are derived under appropriate conditions. Depending on the underlying stochastics, maximizing random measures can be absolutely continuous, discrete, and also singular.

In the microeconomic context, it is natural to embed the above maximization problem in an equilibrium framework. In the last part of this thesis, we give a general existence result for such an equilibrium.

To my parents.

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Introduction

In this thesis, we study the problem of maximizing certain concave functionals on the space of optional random measures. Such functionals arise naturally in microeconomic theory where their maximization corresponds to finding the optimal consumption plan of some economic agent with non–additive intertemporal preferences.

This optimization problem has been addressed before in Hindy, Huang, and Kreps (1992), Hindy and Huang (1993), and Benth, Karlsen, and Reikvam (1999). These authors specify a Markovian framework which permits an application of the well–known methods of Dynamic Programming. We develop an alternative approach which allows us to clarify the structure of maximizing measures in a general semimartingale setting.

In this new approach, we first characterize the maximizing measure by an infinite–dimensional version of the Kuhn–Tucker Theorem. In a second step, we use the first–order conditions derived in this theorem to reduce the optimization problem to a new kind of representation problem for optional processes. Its solution yields a stochastic reference process from which the maximizing random measure can be reconstructed explicitly. In this sense, the representation problem serves as a non–Markovian substitute for the Hamilton–Jacobi–Bellman equation of Dynamic Programming.

Moreover, the representation problem is of independent analytical and probabilistic interest. In the special deterministic case, its solution is constructed in terms of certain envelopes which exhibit a specially developed time–inhomogeneous form of convexity. The construction of a solution in the general stochastic case rests upon an intimate relation to a non–standard optimal stopping problem which also arises in the context of Gittins' problem of optimal dynamic scheduling. In fact, the existence proof for a solution to our stochastic representation problem uses techniques introduced by El Karoui and Karatzas (1994) in their fundamental study of the Gittins index.

Under appropriate conditions, we derive closed–form solutions to our problem. Depending on the underlying stochastics, a whole variety of maximizing random measures occurs. In fact, optimal measures can be absolutely continuous, discrete, or even singular.

In the microeconomic context, it is natural to embed the above maximization problem in an equilibrium framework. In the last part of this thesis, we give a general existence result for such an equilibrium on our space of optional random measures.

Before we discuss the mathematical results of this thesis in greater detail, let us first describe its economic background and motivation.

Economic Motivation

Since its path–breaking treatment by Merton (1969), the problem of optimal consumption and investment in a financial market has become a very active part of both microeconomic theory and the mathematical fields of stochastic calculus and optimization. From an economic point of view, this problem is a key topic whose practical relevance is underlined by the increasing influence of financial markets on global economic development. On a theoretical level, it combines the classic approaches to decision–making under uncertainty with the theory of dynamic asset pricing. In a mathematical context, this optimization problem has become a source of many interesting problems at the interface of convex analysis, martingale theory, and stochastic calculus.

In order to illustrate the influence of the investment–consumption problem on economic and mathematical research in more detail, let us briefly review its history during the last thirty years. Following Samuelson (1964), Merton (1969, 1971) models the financial price fluctuation of asset prices by geometric Brownian motion. The financial market is supposed to be frictionless and trading takes place in continuous time. As a mathematical specification of preferences on intertemporal consumption patterns, Merton chooses a time–additive extension of von Neumann–Morgenstern's expected utility. Using techniques from Dynamic Programming, he derives the Hamilton–Jacobi–Bellman equation for the thus specified utility maximization problem and solves it explicitly for some special utility functions.

With the growing use of martingale theory in finance — initiated by the seminal papers Harrison and Kreps (1979) and Harrison and Pliska (1981) — a second approach emerged: the so–called martingale method. Initially developed by Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989), this approach uses a separation of the problem into two tasks. The first task is to determine the optimal consumption policy when prices for consumption are given by a complete set of Arrow–Debreu forward prices. This task essentially amounts to a static optimization problem which can be addressed by methods of convex analysis set forth, e.g., in Bismut (1973). The second task consists in calculating the investment strategy which finances the previously obtained optimal policy. The solution of this problem involves the celebrated results by Black and Scholes (1973) and Merton (1973) on dynamic asset pricing by no–arbitrage principles.

The fundamental economic insights of Black, Merton, and Scholes triggered a remarkable number of beautiful mathematical discoveries which have enriched especially martingale theory and stochastic calculus. One outstanding example is the characterization of absence of arbitrage by existence of an equivalent martingale measure. First formulated in Harrison and Kreps (1979), this equivalence was subsequently generalized in a long series of papers including Dalang, Morton, and Willinger (1990), Delbaen and Schachermayer (1994) and Delbaen and Schachermayer (1998). Another important example is the duality theory between the set of attainable contingent claims and the set of martingale measures. For complete markets the duality reduces to uniqueness of the equivalent martingale measure, a result already included in Harrison and Kreps (1979). For incomplete markets the first continuous–time account is El Karoui and Quenez (1995) whose result was subsequently generalized and extended by Kramkov (1996), Föllmer and Kramkov (1997), and Föllmer and Kabanov (1998).

The mathematical results sketched above allowed to generalize Merton's original asset price model from geometric Brownian motion to a general semimartingale setting. In addition, portfolio constraints were addressed (see, e.g., Cvitanic and Karatzas (1992)), higher interest rates for borrowing than lending were considered (cf., e.g., Cuoco and Cvitanic (1998)), and even market frictions in form of transaction costs were incorporated (see, e.g., Shreve and Soner (1994), Cvitanic and Karatzas (1996)). Thus, as far as the underlying financial market model is concerned, a remarkably far–reaching mathematical generalization of Merton's original model has been achieved.

However, the situation is much less satisfactory with the other part of Merton's model, namely the mathematical specification of intertemporal preferences. Indeed, Merton's time–additive extension of von Neumann–Morgenstern's (1944) expected utility is still *the* standard model for preferences on consumption streams — despite a number of severe economic objections against such a specification of preferences.

In the first place, the classic critique on von Neumann–Morgenstern's expected utility and their implicit independence axiom also applies to Merton's extension of this preference concept; see Savage (1954) and Anscombe and Aumann (1963) for a remedy based on their concept of subjective probabilities, and Arrow (1953), Debreu (1959) for their solution via state–dependent utilities. There is also empirical evidence against Merton's model, most notably stated in form of Mehra and Prescott's (1985) equity premium puzzle; cf. Constantinides (1990) for an account involving habit formation. Moreover, it has been criticized that the curvature of Merton's utility function must capture simultaneously both the agent's risk aversion and the intertemporal elasticity of substitution in consumption; see Kreps and Porteus (1978) and Duffie and Epstein (1992) for, respectively, a discrete–time and a continuous–time model which allows to disentangle these two preference characteristics.

The most fundamental caveat was raised by Hindy, Huang, and Kreps (1992). Their critique focuses on the very basis of continuous–time preference theory and applies not only to Merton's time–additive extension of von Neumann–Morgenstern utility but actually to all the previously mentioned preference specifications. The central point of their critique is that, concerning slight shifts of consumption in time, the presented preference models are not as robust as one would expect preferences of 'real' economic agents to be.

This point can be illustrated easily in the standard time–additive setting by examining the induced intertemporal substitution properties. Assume, for instance, that having a good meal is modeled by a certain constant rate of consumption for one hour. Now, compare the consumption plan of having such a meal once every day with the plan to have seven such meals from morning to evening on one day and no more meals for the rest of the week. One can hardly doubt that, due to obvious substitution effects, real economic agents would prefer the first plan to the latter. In the time–additive setting, however, both consumption plans will yield essentially the same utility as every single meal contributes to total utility separately. In other words, the standard setting exhibits complementarity of consumption over time rather than local substitutability of consumption.

Essentially, the preceding reasoning also applies for the modifications of the standard preference model mentioned above. In fact, Hindy, Huang, and Kreps argue that agents are indifferent between slight alterations of a consumption plan in both the amounts consumed at every time and the timing of the whole plan. Mathematically, this economic kind of closeness between consumption plans is captured by the Prohorov–distance between nonnegative, finite measures on some time interval. Therefore, utility functionals should be continuous with respect to this distance. However, as Hindy, Huang, and Kreps prove, any utility functional which directly depends on consumption rates in a non–linear way cannot have this economically desirable continuity property. The intuitive reason is that the rate of consumption reacts too sensitively to small changes of the consumption plan. Therefore, the standard specifications of intertemporal preferences have to be rejected from an economic point of view.

As an example for a utility functional which exhibits the economically indicated robustness, Hindy, Huang, and Kreps propose to replace the rate of consumption in Merton's original model with some weighted average of past consumption. This average is interpreted as the level of satisfaction which the agent derives from consumption.

Once this new approach to intertemporal choice theory has been accepted, it is important to understand the consumption behavior which is induced by such Hindy–Huang–Kreps preferences. The first to analyze this question were Hindy, Huang, and Kreps themselves in their 1992 paper which treats a special deterministic case. The paper Hindy and Huang (1993) extends the solution to the framework of geometric Brownian motion. In a recent paper by Benth, Karlsen, and Reikvam (1999), the problem is solved in a setting where stock prices are driven by a Lévy process.

It is a central aim of the present thesis to clarify the structure of optimal consumption patterns in a general semimartingale setting which extends beyond the Markovian framework. Our analysis will lead us to an infinite dimensional version of the Kuhn–

Tucker Theorem and a new type of stochastic representation problem. Under certainty, this problem will be solved by means of a specially defined time–inhomogeneous notion of convexity. The general stochastic version of this problem will be related to the theory of the Gittins–index in the problem of optimal dynamic scheduling.

We are now going to describe our mathematical results in greater detail.

Mathematical Results

Chapter 1 sets up the basic framework for our subsequent studies of intertemporal consumption choice. We follow in spirit, but on a slightly more general level, the approach proposed in Hindy, Huang, and Kreps (1992) and Hindy and Huang (1992).

As a mathematical description of intertemporal consumption patterns we choose the set $\mathcal{C}$ of nonnegative, finite, optional random measures on some time interval. This consumption space is endowed with the topology of weak convergence in probability. Due to the metrization of this topology via the Prohorov–distance, continuous preferences on this consumption space have the economically indicated robust substitution property.

We specify a general class of state–dependent utilities U which have this property and which, in addition, are monotone and concave and have a subgradient $\nabla U(C)$ at each point $C \in \mathcal{C}$. The supporting property of these subgradients will be a central mathematical tool throughout this thesis. Proposition 1.4 shows how these conditions on U carry over to its expectation $V = \mathbb{E}U(.)$. Finally, we verify that our benchmark example of Hindy–Huang–Kreps preferences fits into the described framework.

Within this framework, Chapter 2 studies the problem of optimal consumption and investment choice for a single economic agent who acts as a price–taker on a financial market. Mathematically, this amounts to maximizing a concave functional on the set of optional random measures subject to certain linear constraints. Adapting techniques from Cuoco (1997), Theorem 2.1 provides a general uniqueness and existence result for this problem. It covers both complete and incomplete financial markets, allowing even for convex portfolio constraints. In our approach, the main tools are Komlós' (1967) compactness principle for L^1–bounded sequences of random variables and the duality of hedgeable claims and martingale measures as described in Föllmer and Kramkov (1997).

We proceed by investigating the structure of the optimal consumption plan when markets are complete. As has been shown in, e.g., Hindy and Huang (1993) the Hindy–Huang–Kreps utility maximization problem falls into the class of stochastic singular control problems. A central topic of this thesis consists in the mathematical analysis of this singular control problem in a general semimartingale setting. We do not follow the usual Dynamic Programming Approach since its main tool, the Hamilton–Jacobi–Bellman equation, necessarily requires a Markovian framework. Instead, using the concavity of the problem, we prove an infinite–dimensional version of the Kuhn–Tucker Theorem. The corresponding Theorem 2.2 characterizes the optimal consumption plan $C^* \in \mathcal{C}$ by the necessary and sufficient first–order conditions

$$\nabla V(C^*) \leq M\psi \quad \text{and} \quad \nabla V(C^*) = M\psi \;\; dC^*\text{–a.e.}$$

Here, $M > 0$ is a Lagrange multiplier and ψ denotes the unique state–price density of the considered complete financial market. This characterization is valid for any state–dependent utility conforming to the general framework of Chapter 1. In contrast to its time–additive pendant, it does not yield immediately a description of the optimal plan. Indeed, inverting the second first–order condition in order to solve for C^* — as done in the time–additive framework — is no longer appropriate in our setting since its singular nature entails that the equality $\nabla V(C^*) = M\psi$ might almost surely almost never hold true. As this standard approach fails, it seems difficult in general to obtain additional information about the optimal plan from the above first–order conditions. For the special case of Hindy–Huang–Kreps utilities, however, Theorem 2.3 shows how these conditions can be used to determine the optimal plan in a systematic way.

The main idea is to reduce this optimization to a new kind of stochastic representation problem which will be explained below. The key concept characterized by this representation problem is a stochastic process which we call the 'minimal level of satisfaction'. The optimal consumption plan can be reconstructed explicitly in terms of this minimal level process. Loosely speaking, the investor should optimally consume 'just enough' to ensure that his level of satisfaction never falls below this minimal level. In this sense, the representation problem can be viewed as a substitute for the Hamilton–Jacobi–Bellman equation in our non–Marovian setting.

Chapter 3 treats the representation problem characterizing the minimal level of satisfaction. In its general formulation, the problem consists in the construction of a progressively measurable process L such that a given optional process X with $X(\hat{T}) = 0$ can be represented in the form

$$X(s) = \mathbb{E}\left[\int_s^{\hat{T}} f(t, \sup_{s \leq v \leq t} L(v))\, dt \,\middle|\, \mathcal{F}_s \right] \qquad (0 \leq s \leq \hat{T})$$

where f is a given time–inhomogeneous, strictly monotone function.

Theorem 3.1 establishes uniqueness of such a process L up to upper–rightcontinuous modifications and optional sections. In the simplest case when $f(t, l) \equiv -l$, it identifies the solution L — granted there is some — as a progressively measurable version of

$$L(t) = \operatorname*{ess\,inf}_{T} \frac{\mathbb{E}[\, X(T) - X(t) \,|\, \mathcal{F}_t]}{\mathbb{E}[\, T - t \,|\, \mathcal{F}_t]} \qquad (0 \leq t < \hat{T})$$

where the infimum is taken over all stopping times T taking values in $(t, \hat{T}]$.

For a general function f and a deterministic process X, the representation problem can be solved explicitly in terms of a generalized form of convex envelopes. The underlying generalized notion of convexity accounts for the time–inhomogeneity introduced by the function f. Theorem 3.2 and its converse Theorem 3.3 reveal that precisely the lower–semicontinuous functions X with $X(\hat{T}) = 0$ can be represented in the above form when L varies over the deterministic upper–semicontinuous functions.

For existence in the general stochastic case, we follow a suggestion by Nicole El Karoui and relate our representation problem to Gittins' problem of optimal dynamic scheduling under uncertainty as studied in El Karoui and Karatzas (1994). The main idea is to consider a family of auxiliary optimal stopping problems of Gittins–type. The value functions of these optimal stopping problems allow us to construct the solution to our original representation problem.

In Chapter 4, we provide explicit solutions to the Hindy–Huang–Kreps utility maximization problem, using our key concept of the minimal level of satisfaction.

In the case of certainty, it turns out that, if the investor is not 'too impatient', the representation problem can be solved explicitly for a large class of utilities under a finite time horizon by using our previously introduced concept of inhomogeneously convex envelopes. Theorem 4.1 yields the economically intuitive result that an investor with a low initial level of satisfaction immediately starts consuming by taking an initial gulp, whereas a high initial level of satisfaction induces him to wait for a while. After that consumption occurs at rates until from some time on the investor refrains from consuming totally. This behavior is rational for a Hindy–Huang–Kreps utility maximizer since — in contrast to his time–additive counterpart — he obtains utility from past consumption rather than from current consumption alone. This explicit solution extends the results obtained by Hindy, Huang, and Kreps (1992) who derive closed–form solutions for homogeneous utilities with infinite time–horizon. For such utilities, Theorem 4.2 provides the complete solution for all possible choices of the model parameters.

To study the case of uncertainty, we consider a homogeneous setting where consumption prices ψ follow a geometric Lévy process and where the agent's Hindy–Huang–Kreps utility is based on a power–felicity function with infinite time horizon. In this homogeneous framework, Theorem 4.3 describes the optimal consumption plan. The Wiener–Hopf factorization enables us in Theorem 4.4 to characterize explicitly the parameter values for which the utility maximization is well–posed. For Lévy processes without upward jumps, results from fluctuation theory allow us to describe in detail the dual relation between Lagrange multipliers and different amounts of initial wealth. The great structural variety of the derived optimal plans covers the 'standard' form of consumption at rates as well as consumption in gulps or in singular form. This illustrates the flexibility of both the Hindy–Huang–Kreps framework and of our minimal–level–method to address the associated singular control problem.

In our final Chapter 5, we consider a stochastic pure–exchange economy with a finite number of agents whose utilities conform to the Hindy–Huang–Kreps kind of intertemporal substitution specified in Chapter 1. For such an economy, Theorem 5.1 establishes existence of Arrow–Debreu equilibria under fairly general conditions.

As usual in the context of infinite dimensional commodity spaces, the Negishi–method is the basis for the proof of existence. However, in contrast to the usual approach as described, e.g., in Mas-Colell and Richard (1991), we do not restrict price functionals a priori to be continuous on the consumption space. This continuity is established only a posteriori in Theorem 5.2 under the additional assumption that the information flow in the economy is quasi–leftcontinuous and that utility gradients are semimartingales with a continuous compensator. Our approach to prove existence of equilibria relies on a Kuhn–Tucker characterization of efficient allocations and on the structure of supporting prices as weighted maxima of utility gradients. A central technical tool is Komlós' (1967) theorem. In its measure–valued version by Kabanov (1999), this result gives us a powerful compactness principle which we use to prove both existence of efficient allocations and their continuous dependence on agents' weights. In conjunction with an argument going back to Bewley (1969), this continuity allows us to prove upper–hemicontinuity of the usual excess utility correspondence. Existence of equilibrium is then obtained by applying Kakutani's fixed point theorem.

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I would also like to thank Nicole El Karoui who pointed out the Gittins–approach in the context of the representation problem discussed in Chapter 3. Further thanks go to Darrel Duffie who suggested to widen the scope of the equilibrium studies in Chapter 5 from the Hindy–Huang–Kreps framework to a more abstract setting. Thanks also to Jean Bertoin for explaining me the usefulness of the Duality Lemma for Lévy processes.

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Chapter 1

Intertemporal Preferences

This chapter sets up the basic framework for our analysis of intertemporal consumption choice which will be carried out in the subsequent chapters. Essentially, this framework consists of two parts: the mathematical description of intertemporal consumption patterns, and the specification of preferences between such patterns in terms of suitable utility functionals. For both tasks, we follow in spirit, but on a slightly more general level, the approach suggested in the fundamental work of Hindy, Huang, and Kreps (1992) and Hindy and Huang (1992).

On a given filtered probability space, consumption patterns will be described by nonnegative, optional random measures on the time axis. This is a very convenient framework from an economic point of view. It allows us to include not only the 'standard' case of absolutely continuous measures where consumption takes place at rates, but also consumption in gulps or in singular form. We endow this space of consumption patterns with the topology of weak convergence in probability. As pointed out by Hindy, Huang, and Kreps (1992), this topology captures the economic axiom of intertemporal substitution which requires that "consumption at one time should be something of a substitute for consumption at other, near–by times".

This specific choice of topology entails the remarkable consequence that the 'standard' mathematical formalization of intertemporal preferences via time–additive von Neumann–Morgenstern utility functionals is no longer appropriate. Indeed, Hindy, Huang, and Kreps (1992) show that these standard functionals exhibit local substitutability of consumption over time only if they are linear. This, however, is incompatible with risk–aversion, another economically relevant feature of preferences under uncertainty.

Hence, in order to incorporate both substitutability of consumption and risk aversion, other utility functionals have to be considered. In Section 1.1, we introduce a general class of preferences given by expected utility functionals of the form $V(C) = \mathbb{E}U(C)$ where $\mathbb{E}$ is the expectation associated to the underlying probability measure and U is continuous with respect to the weak topology. This continuity ensures that preferences exhibit the desired local substitution of consumption over time. We furthermore require monotonicity and concavity of U, as well as existence of a subgradient $\nabla U(C)$ at each point $C \in \mathcal{C}$. We show in Proposition 1.4 how these assumptions on U carry over to its expectation V. The resulting properties of the expected utility functional V will serve as our technical basis for the following chapters.

In Section 1.2, we verify that the benchmark example of Hindy–Huang–Kreps preferences fits into our general framework. We also consider an extension of these preferences proposed by Hindy, Huang, and Zhu (1997) which captures the effect of habit formation.

1.1 The General Framework

In this section, we describe a general setting for intertemporal preferences in continuous time, following essentially the approach suggested by Hindy, Huang, and Kreps (1992).

1.1.1 The Set of Consumption Patterns

Under **certainty,** the natural space of intertemporal consumption patterns over a fixed time period is given by the set of all nonnegative, finite Borel–measures on some time inteval $[0, \hat{T}]$. Identifying each such measure with its cumulative distribution function, we therefore introduce

$$\mathcal{M}_+ \stackrel{\Delta}{=} \left\{ C : [0, \hat{T}] \to \mathbb{R}_+ \;\middle|\; C \text{ increasing and rightcontinuous} \right\} .$$

As pointed out by Hindy, Huang, and Kreps (1992), the economically appropriate notion of distance on this consumption space is given by the Prohorov–metric

$$d_{\mathcal{M}_+}(C, C') \stackrel{\Delta}{=} \inf \left\{ \varepsilon > 0 \;\middle|\; C((t-\varepsilon) \vee 0) - \varepsilon \leq C'(t) \leq C((t+\varepsilon) \wedge \hat{T}) + \varepsilon \;\forall\, t \in [0, \hat{T}] \right\} .$$

In this distance, small perturbations of a given consumption plan both in size and time are still close to the original plan; see Figure 1.1. It thus captures the economically intuitive feature that preferences are robust not only with respect to slight changes of the amounts consumed at a given time, but also with respect to slight shifts of consumption in time.

The metric $d_{\mathcal{M}_+}$ endows $\mathcal{M}_+$ with the weak*–topology of the pairing $(C[0, \hat{T}], \mathcal{M})$ where $\mathcal{M} \stackrel{\Delta}{=} \mathcal{M}_+ - \mathcal{M}_+$. The corresponding bracket operator $(.,.)$ is

$$(\phi, C) \stackrel{\Delta}{=} \int_0^{\hat{T}} \phi(t)\, dC(t) \qquad \text{for} \quad \phi \in C[0, \hat{T}],\ C \in \mathcal{M} .$$

Convention *Throughout this thesis, integration over time intervals is carried out including the involved finite boundaries. We let any consumption stream C start in $C(0-) \stackrel{\Delta}{=} 0$;*

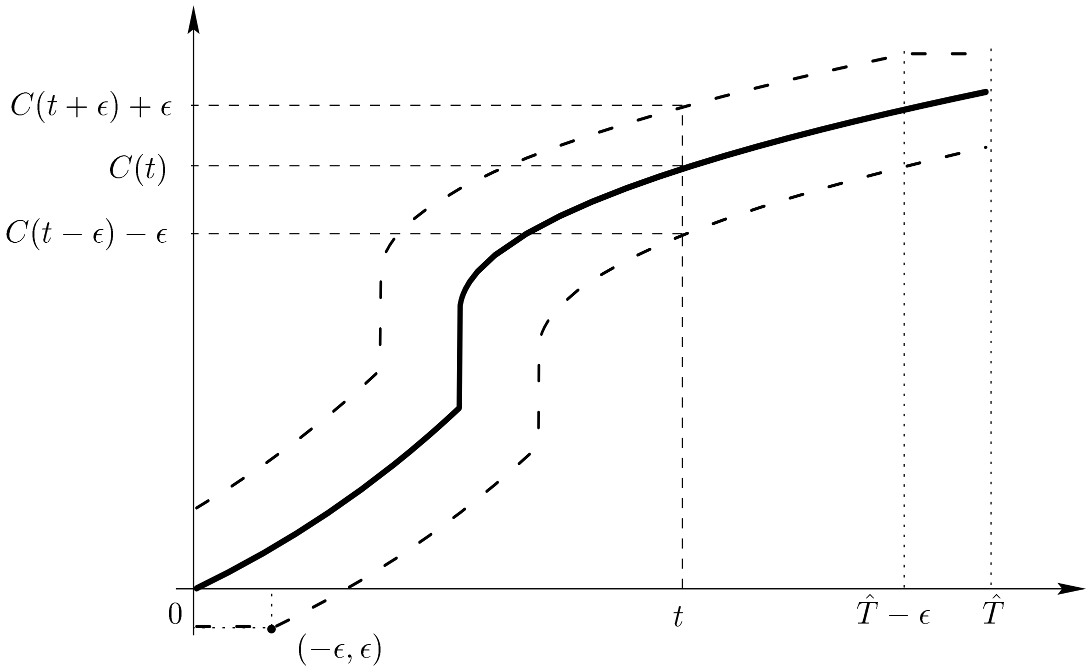

Figure 1.1: A neighborhood of a consumption plan C in the Prohorov–metric. The plans within ε–distance to C are those increasing functions whose graph is completely within the 'sleeve' indicated by the dashed lines.

a positive value at time 0 *indicates an initial consumption gulp and corresponds to a point mass* $C(0) > 0$ *of the measure* dC *at time* $t = 0$*. Similarly, we assume that any other integrator* B *starts from some initial value* $B(0-)$*, which is supposed to be zero unless otherwise stated.*

We recall the natural ordering on $\mathcal{M}_+$ which is given by

$$C \preceq C' \quad \Leftrightarrow \quad C - C' \in \mathcal{M}_+ \, .$$

Remark 1.1 *Note that* $\preceq$ *does not denote the preference relation on consumption patterns which will be used in the sequel. Throughout this thesis, preferences for consumption will be described by an inequality* $\leq$ *between certain utilities. Hence, no confusion should arise.*

Let us now assume that **uncertainty** is modeled by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t,\ 0 \leq t \leq \hat{T}))$ satisfying the usual conditions of rightcontinuity and completeness. We suppose that $\mathcal{F}_0$ is $\mathbb{P}$–a.s. trivial. The filtration $\mathbb{F}$ describes the information flow of the economy.

Under uncertainty, the natural consumption space is

$$\mathcal{C} \triangleq \left\{ C : \Omega \to \mathcal{M}_+ \;\middle|\; (C(t),\ t \in [0, \hat{T}]) \text{ is an adapted process} \right\} .$$

This space can be identified with the set of all finite, nonnegative, optional random measures on $[0, \hat{T}]$. The infinitesimal quantity $dC(\omega, t)$ is interpreted as the amount consumed by the agent at time $t \in [0, \hat{T}]$ if he follows the consumption plan C and if the 'state of the world' is $\omega \in \Omega$. Optionality of the random measure dC corresponds to the condition that the increasing process $(C(t),\ t \in [0, \hat{T}])$ is adapted to the filtration $\mathbb{F}$. In economic terms, this condition means that the agent takes his consumption decision based only on the publicly available information.

A natural extension of the Prohorov–metric $d_{\mathcal{M}_+}$ to the uncertain framework is given by

$$d_{\mathcal{C}}(C, C') \triangleq \mathbb{E}\left[d_{\mathcal{M}_+}(C, C') \wedge 1\right] \quad (C, C' \in \mathcal{C}).$$

This metric endows our consumption space $\mathcal{C}$ with the topology of weak convergence in probability.

Remark 1.2 *Here, we slightly deviate from the setting proposed by Hindy and Huang (1992). Instead of the metric $d_{\mathcal{C}}$, these authors consider distances such as*

$$\|C - C'\|_{\mathcal{C}} \triangleq \mathbb{E}\|C - C'\|_{\mathcal{M}_+} \quad \textit{where} \quad \|C - C'\|_{\mathcal{M}_+} = \int_0^{\hat{T}} |C(t) - C'(t)|\, dt + |C(\hat{T}) - C'(\hat{T})|\,.$$

On the cone $\mathcal{M}_+$ both $\|.\|_{\mathcal{M}_+}$ and $d_{\mathcal{M}_+}$ induce the weak–topology. The distances $\|. - .\|_{\mathcal{C}}$ and $d_{\mathcal{C}}$, however, induce differing topologies on the set of random consumption patterns $\mathcal{C}$ since L^1–convergence merely implies, but is not equivalent to convergence in probability.*

The main reason for this minor deviation from the original Hindy–Huang setting is that the solution to the single agent–utility maximization problem, which we are going to study later, does not necessarily exhibit the $\mathbb{P}$–integrability required by $\|.\|_{\mathcal{C}}$. Thus, restricting the consumption space $\mathcal{C}$ to those plans C for which $\|C\|_{\mathcal{C}} < +\infty$ might rule out this solution.

Similarly, we extend the deterministic bracket operation $(.,.)$ to the case of uncertainty by defining

$$\langle \phi, C \rangle \triangleq \mathbb{E} \int_0^{\hat{T}} \phi(t)\, dC(t)\,.$$

It is well–defined (possibly infinite) for any $C \in \mathcal{C}$ if ϕ is $\mathcal{F} \otimes \mathcal{B}[0, \hat{T}]$–measurable and nonnegative. Passing from such a ϕ to its optional projection ${}^o\phi$ preserves its bracket with any finite optional random measure:

$$\langle \phi, C \rangle = \langle {}^o\phi, C \rangle \quad \text{for all} \quad C \in \mathcal{C}\,; \tag{1.1}$$

see, e.g., Théorème (1.33) in Jacod (1979).

The ordering $\preceq$ naturally extends from $\mathcal{M}_+$ to $\mathcal{C}$ via

$$C \preceq C' \quad \Leftrightarrow \quad C - C' \in \mathcal{C}\,.$$

1.1.2 Utility Functionals

Let us assume that, for every single 'state of the world' $\omega \in \Omega$, an economic agent has specified his preferences between consumption plans in $\mathcal{M}_+$ via a state–dependent utility functional

$$U : \Omega \times \mathcal{M}_+ \to \mathbb{R}\,.$$

Assumption 1.1 *The mapping $U : \Omega \times \mathcal{M}_+ \to \mathbb{R}$ is product–measurable with $U(0) \in L^1(\mathbb{P})$, and there is a set $\Omega^* \in \mathcal{F}$ with full measure $\mathbb{P}[\Omega^*] = 1$ such that the following properties (i)–(iii) hold true:*

(i) For every $\omega \in \Omega^*$, the mapping $U_\omega : (\mathcal{M}_+, d_{\mathcal{M}_+}) \to \mathbb{R}$ is continuous, strictly concave, and strictly increasing with respect to the natural ordering $\preceq$ on $\mathcal{M}_+$.

(ii) For each consumption plan $C \in \mathcal{M}_+$, there is an $\mathcal{F}$–measurable random variable $\nabla U(C)$ taking values in the set of continuous functions $C[0, \hat{T}]$ which defines a subgradient of U in the sense that on Ω^* we have

$$U(C') - U(C) \leq (\nabla U(C), C' - C) \quad \textit{for all} \quad C' \in \mathcal{M}_+\,.$$

(iii) The above subgradient gives rise to a continuous mapping

$$\nabla U_\omega : (\mathcal{M}_+, d_{\mathcal{M}_+}) \to C[0, \hat{T}]$$

for any $\omega \in \Omega^*$ where we endow $C[0, \hat{T}]$ with its weak topology $\sigma(C[0, \hat{T}], \mathcal{M})$.

The continuity of U with respect to the Prohorov–metric $d_{\mathcal{M}_+}$ ensures that the induced preferences satisfy Hindy–Huang–Kreps' 'axiom of intertemporal substitution'. Apart from that, the preceding assumption essentially requires monotonicity and convexity of preferences and a sufficient degree of smoothness. Analogous assumptions on state–dependent utilities are usually made when one considers agents who derive their utility from terminal wealth only; see, e.g., Mas-Colell and Zame (1991) and Föllmer and Leukert (2000).

A first consequence of this assumption is

Proposition 1.3 *The subgradient ∇U introduced by Assumption 1.1 is uniquely determined in the sense that on Ω^* we have*

$$\nabla U(C)(t) = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon = 0+} U\left(C + \varepsilon 1_{[t, \hat{T}]}\right) > 0$$

for all $t \in [0, \hat{T}]$ and all $C \in \mathcal{M}_+$.

PROOF : Fix $\omega \in \Omega^*$, $t \in [0, \hat{T}]$, and $C \in \mathcal{M}_+$. Put $f(\varepsilon) \stackrel{\Delta}{=} U_\omega\left(C + \varepsilon 1_{[t,\hat{T}]}\right)$ $(\varepsilon \geq 0)$. Due to Assumption 1.1, this function is continuous and concave on $[0, +\infty)$. Thus, the righthand derivative $\partial^+ f$ exists, and we have to show $\partial^+ f(0) = \nabla U_\omega(C)(t)$.

For this, note first that the strict monotonicity of U and the subgradient property of $\nabla U_\omega(C)$ imply

$$0 < f(\varepsilon) - f(0) \leq \left(\nabla U_\omega(C), \varepsilon 1_{[t,\hat{T}]}\right) = \varepsilon \nabla U_\omega(C)(t)\,.$$

Therefore, $\nabla U_\omega(C)(t)$ is strictly positive with

$$\nabla U_\omega(C)(t) \geq \frac{f(\varepsilon) - f(0)}{\varepsilon} \to \partial^+ f(0) \quad \text{as} \quad \varepsilon \downarrow 0\,.$$

In order to prove the converse inequality, let $\eta \geq \varepsilon > 0$ and use the subgradient property of $\nabla U_\omega\left(C + \eta 1_{[t,\hat{T}]}\right)$ for the estimate

$$f(\eta - \varepsilon) - f(\eta) \leq -\varepsilon\left(\nabla U_\omega\left(C + \eta 1_{[t,\hat{T}]}\right), 1_{[t,\hat{T}]}\right)\,.$$

Divide by $-\varepsilon < 0$ and then let $\varepsilon \downarrow 0$ to obtain

$$\partial^- f(\eta) \geq \left(\nabla U_\omega\left(C + \eta 1_{[t,\hat{T}]}\right), 1_{[t,\hat{T}]}\right)\,.$$

For $\eta \downarrow 0$, the left side of this inequality tends to $\partial^+ f(0)$ while its right side has the limit

$$\left(\nabla U_\omega(C), 1_{[t,\hat{T}]}\right) = \nabla U_\omega(C)(t)$$

by weak continuity of $\nabla U_\omega(.)$ (Assumption 1.1 (iii)). Hence, $\partial^+ f(0) \geq \nabla U_\omega(C)(t)$ which is the desired converse inequality. □

Given a functional U satisfying Assumption 1.1, we obtain preferences on our space $\mathcal{C}$ of uncertain consumption patterns via the expected utility functional

$$V(C) \stackrel{\Delta}{=} \mathbb{E}U(C) \quad (C \in \mathcal{C})\,.$$

This functional inherits the following properties:

Proposition 1.4 *(i) The mapping $V : \mathcal{C} \to \mathbb{R} \cup \{+\infty\}$ is concave and increasing with respect to the ordering $\preceq$. These properties hold true in their strict sense on the domain*

$$\mathrm{Dom}(V) \stackrel{\Delta}{=} \{C \in \mathcal{C} \mid V(C) \in \mathbb{R}\}\,.$$

The mapping V is continuous with respect to the metric $d_{\mathcal{C}}$ on every subset $\mathcal{C}' \subset \mathcal{C}$ for which the family $(U(C),\ C \in \mathcal{C}')$ is uniformly integrable.

(ii) *The optional projection* $\nabla V(C) \stackrel{\Delta}{=} {}^o\nabla U(C) \geq 0$ *defines a subgradient for* V *in the sense that, for* $C \in \mathrm{Dom}(V)$*, we have*

$$V(C') - V(C) \leq \langle \nabla V(C), C' - C \rangle \in (-\infty, +\infty] \quad \textit{for all } C' \in \mathcal{C}\,.$$

(iii) *The subgradient* ∇V *induces a lower–semicontinuous mapping in the sense that*

$$\liminf_n \langle \nabla V(C^n), C \rangle \geq \left\langle \nabla V(C^0), C \right\rangle \quad \textit{for all } C \in \mathcal{C}$$

whenever $(C^n,\ n = 1, 2, \dots)$ *is a sequence of consumption plans converging to* C^0 *in* $(\mathcal{C}, d_{\mathcal{C}})$*.*

Proof :

(i) Concavity and monotonicity are inherited from U. In order to prove continuity, let $C^n \in \mathcal{C}' \subset \mathcal{C}$ $(n = 1, 2, \dots)$ converge to C^0 in the metric $d_{\mathcal{C}}$, i.e., let C^n tend to C^0 weakly in probability. In connection with the standard subsequence argument, Assumption 1.1 (i) yields that $U(C^n)$ converges to $U(C^0)$ in probability. Since $(U(C), C \in \mathcal{C}')$ is uniformly integrable by assumption, Lebesgue's theorem gives convergence even in $L^1(\mathbb{P})$ and we are done.

(ii) Note first that by monotonicity of U we have $\nabla U(C) \geq 0$ $\mathbb{P}$–a.s., and thus also $\nabla V(C) = {}^o\nabla U(C) \geq 0$ for all $C \in \mathcal{C}$. Hence, formula (1.1) yields

$$\langle \nabla V(C), C' \rangle = \mathbb{E}\left(\nabla U(C), C'\right) \quad \text{for any} \quad C, C' \in \mathcal{C}\,.$$

For $C' = C \in \mathrm{Dom}(V)$, this implies

$$0 \leq \langle \nabla V(C), C \rangle = \mathbb{E}\left(\nabla U(C), C\right) \leq \mathbb{E}\left[U(C) - U(0)\right] = V(C) - V(0) \qquad (1.2)$$

where, for the second estimate, we used the subgradient property of $\nabla U(C)$ (Assumption 1.1 (ii)). From the same property, we infer

$$U(C') - U(C) \leq \left(\nabla U(C), C' - C\right) \quad \mathbb{P}\text{–a.s.}\,.$$

For $C \in \mathrm{Dom}(V)$, the expectation of the left side in this inequality is well–defined in $(-\infty, +\infty]$. Our estimate (1.2) shows that we may also take expectations on the right side of this inequality. This proves assertion (ii).

(iii) Fix $C \in \mathcal{C}$ and assume that C^n $(n = 1, 2, \dots)$ converges to C^0 in $(\mathcal{C}, d_{\mathcal{C}})$. Let n' be a subsequence realizing the lim inf in assertion (iii). Passing to a further subsequence n'', we may assume that $C^{n''}$ almost surely converges to C^0 in $(\mathcal{M}_+, d_{\mathcal{M}+})$. By Assumption 1.1 (iii), this yields

$$\left(\nabla U(C^{n''}), C\right) \to \left(\nabla U(C^0), C\right) \quad \mathbb{P}\text{–a.s.}$$

As all these random variables are nonnegative, Fatou's lemma allows us to deduce

$$\liminf_n \langle \nabla V(C^n), C\rangle = \lim_{n''} \mathbb{E}\left(\nabla U(C^{n''}), C\right) \geq \mathbb{E}\left(\nabla U(C^0), C\right) = \left\langle \nabla V(C^0), C\right\rangle .$$

□

The preceding Proposition 1.4 provides the technical basis for our study of the utility maximization problem and for a general equilibrium theory in terms of preferences with intertemporal substitution.

1.2 Discussion and Examples

Let us next consider some key examples of utility functionals and check whether they fit into the above framework or not.

1.2.1 Standard Time–Additive Utilities

Standard time–additive utilities are based on the *rate* of consumption and are thus defined on the smaller consumption space

$$\tilde{\mathcal{C}} \triangleq \{C \in \mathcal{C} \mid dC \text{ absolutely continuous with respect to Lebesgue measure}\} .$$

They take the von Neumann–Morgenstern form

$$\tilde{V}(C) = \mathbb{E}\tilde{U}(C) = \mathbb{E}\int_0^{\hat{T}} u(t, \dot{C}(t))\, dt \quad (C \in \tilde{\mathcal{C}})$$

where $\dot{C}$ denotes the Lebesgue–density of dC. The function $u : [0, \hat{T}] \times [0, +\infty) \to \mathbb{R}$ is called felicity function and is usually assumed to be increasing and concave in its second argument.

Remark 1.5 *More generally, one may account for effects like habit formation or addiction by letting instantaneous felicity not only depend on the current rate of consumption but also on an index of past consumption; see, e.g., Constantinides (1990) and Schroder and Skiadas (1998). Compare also the end of the following section where we discuss a preference model with habit formation proposed by Hindy, Huang, and Zhu (1997).*

While this setting has been studied extensively in the literature, it suffers from several severe deficiencies from an economic point of view. Indeed, in the first place, it does not allow to work with the economically natural consumption space of optional random measures $\mathcal{C}$. Thus, it excludes possibly relevant phenomena such as consumption in gulps or consumption in singular form.

Second, it lacks a natural discrete–time analogue which would sustain the form of the utility functional by a limiting procedure. This deficiency is mainly due to the fact that, in the standard setting, instantaneous felicity is obtained from the current *speed* of consumption, a quantity which does not allow a consistent translation to discrete time whenever felicity is strictly concave in its second argument.

As already pointed out in the introduction to this thesis, the third and probably most severe deficiency is that standard von Neumann–Morgenstern preferences do *not* allow for intertemporal substitution of consumption with the *economically indicated robustness*, unless they are linear. This has been shown by Hindy, Huang, and Kreps (1992) for the case of certainty (see their Proposition 6) and by Hindy and Huang (1992) (Proposition 8) for the uncertain framework.

Thus, standard von Neumann–Morgenstern utilities do *not* fit into our framework.

1.2.2 Hindy–Huang–Kreps Utilities

The most prominent preferences fitting into our setting are the so–called

Hindy–Huang–Kreps Preferences. These preferences are induced by utility functionals of the form

$$U(C) \triangleq \int_0^{\hat{T}} u(t, Y(C)(t))\,dt \quad (C \in \mathcal{M}_+) \tag{1.3}$$

where u is some felicity function and where

$$Y(C)(t) \triangleq \eta e^{-\int_0^t \beta(s)\,ds} + \int_0^t \beta(s) e^{-\int_s^t \beta(v)\,dv}\,dC(s) \quad (0 \le t \le \hat{T})\,.$$

Here, $\eta \geq 0$ is a constant and $\beta : [0, \hat{T}] \to (0, +\infty)$ is a strictly positive continuous function. The process $Y(C)$ describes the evolution of the agent's level of satisfaction if he follows the consumption plan $C \in \mathcal{C}$. This level is given as the combination of a declining effect of the agent's initial satisfaction $\eta = Y(C)(0-)$ with an exponentially weighted average of past consumption.

Remark 1.6 *(i) For a given consumption plan $C \in \mathcal{M}_+$, the induced level of satisfaction $Y(C)$ evolves according to the ODE*

$$Y(C)(0-) = \eta, \quad dY(C)(t) = \beta(t)\,(dC(t) - Y(C)(t-)\,dt) \quad (0 \le t \le \hat{T})\,. \tag{1.4}$$

Hence, past consumption will affect future levels of satisfaction only through the induced current *level of satisfaction. This observation will be important for our investigation of the structure of optimal consumption plans in Sections 2.3.2 and 2.3.3.*

(ii) *More generally than* (1.4), *we could assume a level dynamics of the form*

$$Y(C)(0-) = \eta, \quad dY(C)(t) = \beta^1(t)\,dC(t) - \beta^2(t)Y(C)(t-)\,dt \quad (0 \leq t \leq \hat{T})$$

for some adapted, continuous processes β^1, $\beta^2 > 0$. *This would lead to a state–dependent utility functional* U.

(iii) *Instead of considering* $Y(C)(t)$ *as a level of satisfaction derived from consuming a perishable good, one may consider this quantity alternatively as an index for the service flow derived from previously bought, durable goods; see Hindy and Huang (1993).*

We make the following

Assumption 1.2 *The felicity function* $u : [0, \hat{T}] \times [0, +\infty) \to \mathbb{R}$ *is continuous. It is strictly increasing, strictly concave, and differentiable in its second argument* y. *The partial derivative* $\partial_y u$ *is a continuous, real–valued mapping on* $[0, \hat{T}] \times (0, +\infty)$ *and satisfies the Inada–conditions*

$$\partial_y u(t, 0+) = +\infty \quad \text{and} \quad \partial_y u(t, +\infty) = 0 \quad \text{for all} \quad t \in [0, \hat{T}]\,.$$

This assumption allows us to introduce, for any $C \in \mathcal{M}_+$, the function

$$\nabla U(C)(t) \triangleq \int_t^{\hat{T}} \partial_y u(s, Y(C)(s))\beta(t)e^{B(t)-B(s)}\,ds \quad (0 \leq t \leq \hat{T}) \tag{1.5}$$

where, for ease of notation, we put

$$B(t) \triangleq \int_0^t \beta(s)\,ds \quad (0 \leq t \leq \hat{T})\,.$$

Remark 1.7 *The gradient* $\nabla U(C)(t)$ *of* (1.5) *has already been computed by Duffie and Skiadas (1994), see their Example 5.*

The following Proposition shows that indeed ∇U is a subgradient of U.

Proposition 1.8 *Let* u *be a felicity function satisfying Assumption 1.2. Then the Hindy–Huang–Kreps utility functional* (1.3) *has the following properties if the agent's initial level of satisfaction* η *is strictly positive:*

(i) $U : (\mathcal{M}_+, d_{\mathcal{M}_+}) \to \mathbb{R}$ *is continuous, concave, and increasing. The latter two properties hold true in their strict sense on*

$$\{C \in \mathcal{M}_+ \mid \Delta C(\hat{T}) = 0\}\,.$$

(ii) *For any* $C \in \mathcal{M}_+$, $\nabla U(C)$ *is a continuous function taking values in* $[0, +\infty)$. *It satisfies*

$$U(C') - U(C) \leq (\nabla U(C), C' - C) \quad \text{for all} \quad C' \in \mathcal{M}_+ \,. \tag{1.6}$$

(iii) $\nabla U(C)$ *depends on* $C \in \mathcal{M}_+$ *continuously in the strong sense:*

$$\|\nabla U(C^n) - \nabla U(C^0)\|_\infty \to 0 \quad \text{whenever} \quad C^n \to C^0 \quad \text{in } (\mathcal{M}_+, d_{\mathcal{M}_+}).$$

If the initial level of satisfaction is $\eta = 0$, *then property (i) holds true without change, however, properties (ii) and (iii) are weakened to*

(iv) $\nabla U(C)$ *is a measurable function taking values in* $[0, +\infty]$. *The subgradient property* (1.6) *holds true provided* $U(C) > -\infty$.

(v) *We have weak semicontinuity of* $\nabla U(.)$ *in the sense that*

$$\liminf_n \, (\nabla U(C^n), C) \geq \big(\nabla U(C^0), C\big) \quad \text{for all} \quad C \in \mathcal{M}_+$$

whenever C^n *tends to* C^0 *in* $(\mathcal{M}_+, d_{\mathcal{M}_+})$. *If, in addition to* $d_{\mathcal{M}_+}(C^n, C) \to 0$, *we also have* $kC \preceq C^n; (n = 1, 2, \ldots)$ *for some constant* $k > 0$, *then even*

$$(\nabla U(C^n), C) \to \big(\nabla U(C^0), C\big) \quad \text{for all} \quad C \in \mathcal{M}_+$$

holds true.

Proof : For $\eta > 0$, each level of satisfaction $Y(C)$ $(C \in \mathcal{M}_+)$ is bounded away from zero on $[0, \hat{T}]$ by $\eta e^{-B(\hat{T})}$. Thus, the singularity of $\partial_y u$ at $y = 0$ does not matter in this case.

(i) Concavity and monotonicity of U follow from our assumptions on u and from affine linearity and monotonicity of $Y(C)$ in C. The strict form of these properties on $\{C \in \mathcal{M}_+ \mid \Delta C(\hat{T}) = 0\}$ follows from the observation that two different consumption plans from this set will induce levels of satisfaction that differ over a non–trivial time interval.

To prove continuity of U with respect to $d_{\mathcal{M}_+}$, let $C^n \in \mathcal{M}_+$ $(n = 1, 2, \ldots)$ converge weakly to $C^0 \in \mathcal{M}_+$. Note that, for any fixed $t \in \{\Delta C^0 = 0\}$, the function

$$s \mapsto \beta(s) e^{B(s)-B(t)} 1_{[0,t]}(s)$$

is continuous dC^0–a.e. Hence, the Portemanteau Theorem yields

$$\begin{aligned} Y(C^n)(t) &= \eta e^{-B(t)} + \int_0^{\hat{T}} \beta(s) e^{B(s)-B(t)} 1_{[0,t]}(s)\, dC^n(s) \\ &\to \eta e^{-B(t)} + \int_0^{\hat{T}} \beta(s) e^{B(s)-B(t)} 1_{[0,t]}(s)\, dC^0(s) = Y(C^0)(t) \end{aligned}$$

for all such t. In particular, we have that Lebesgue–a.e.

$$u(t, Y(C^n)(t)) \to u(t, Y(C^0)(t)) \quad (n \uparrow +\infty)$$

Moreover, the sequence $(C^n(\hat{T}),\ n = 1, 2, \ldots)$ is bounded by some constant as it converges to $C^0(\hat{T})$ for $n \uparrow +\infty$. Hence, we have

$$u(t, 0) \le u(t, Y(C^n)(t)) \le u(t, \text{const.}) \le \text{const.}\ ,$$

and we may, thus, use dominated convergence to obtain our claim $U(C^n) \to U(C^0)$.

(ii) Condition $\eta > 0$ ensures that $\nabla U(C)(t)$ is real–valued for every $t \in [0, \hat{T}]$. Thus, its continuity in t can immediately be read off (1.5). Now, consider $C, C' \in \mathcal{M}_+$, put $Y \triangleq Y(C)$, $Y' \triangleq Y(C')$, and use the subgradient property of $\partial_y u$ to estimate

$$u(t, Y'(t)) - u(t, Y(t)) \le \partial_y u(t, Y(t))\{Y'(t) - Y(t)\}\,.$$

Using the definition of Y and Y', we may rewrite this as

$$u(t, Y'(t)) - u(t, Y(t)) \le \partial_y u(t, Y(t)) \int_0^t \beta(s) e^{B(s)-B(t)} \{dC'(s) - dC(s)\}\,. \quad (1.7)$$

Integrating with respect to $t \in [0, \hat{T}]$ and using Fubini's theorem yields the claimed subgradient property of $\nabla U(C)$.

(iii) To show continuity of $\nabla U(.)$ in the strong sense, consider $C^n \in \mathcal{M}_+$ $(n = 0, 1, 2, \ldots)$ with $d_{\mathcal{M}_+}(C^n, C^0) \to 0$ and recall from the proof of (i) that $Y(C^n) \to Y(C^0)$ Lebesgue–a.e. in this case. Thus, $\partial_y u(s, Y(C^n)(s)) \to \partial_y u(s, Y(C^n)(s))$ for Lebesgue–a.e. $s \in [0, \hat{T}]$. In addition,

$$0 \le \partial_y u(s, Y(C^n)(s)) \le \partial_y u(s, \eta e^{-B(s)})$$

where the latter quantity is in $L^1(ds)$ since $\eta > 0$. Thus, we may use dominated convergence to obtain

$$|\nabla U(C^n)(t) - \nabla U(C^0)(t)| \le \text{const.} \int_0^{\hat{T}} \left|\partial_y u(s, Y(C^n)(s) - \partial_y u(s, Y(C^0)(s)\right| \, ds \to 0$$

uniformly in $t \in [0, \hat{T}]$ as $n \uparrow +\infty$.

In case $\eta = 0$, the proof of part (i) still goes through. Concerning the gradient properties, however, we have to take care of the singularity of $\partial_y u$ at $y = 0$.

(iv) As $\eta = 0$, the function $\nabla U(C)$ now may take the value $+\infty$. For the subgradient estimate we may assume that $(\nabla U(C), C') < +\infty$ because otherwise the statement becomes trivial (if $(\nabla U(C), C) < +\infty$) or meaningless (if $(\nabla U(C), C) = +\infty$).

Using the (implicit) monotonicity of both $u(t, Y(t))$ and $\partial_y u(t, Y(t))$ in η, the subgradient estimate follows in this case from its validity in case $\eta > 0$ by letting this parameter tend to zero and using monotone convergence.

(v) For (semi-)continuity of $\nabla U(.)$ in the claimed weak sense, use Fubini's theorem to write

$$(\nabla U(C^n), C) = \int_0^{\hat{T}} \partial_y u(s, Y(C^n)(s)) \{Y(C)(s) - Y(0)(s)\} \, ds\,.$$

By the same argument as above, the nonnegative integrand tends to

$$\partial_y u(s, Y(C^0)(s)) \{Y(C)(s) - Y(0)(s)\}$$

for Lebesgue–a.e. $s \in [0, \hat{T}]$. Hence, we may apply Fatou's lemma to deduce the asserted weak semicontinuity.

If, however, $kC \preceq C^n$ for some constant $k > 0$, the integrand in addition satisfies

$$\begin{aligned} 0 &\le \partial_y u(s, Y(C^n)(s)) \{Y(C)(s) - Y(0)(s)\} \\ &\le \partial_y u(s, Y(kC)(s)) \{Y(kC)(s) - Y(0)(s)\} / k \end{aligned}$$

where the latter function is Lebesgue–integrable because

$$\begin{aligned} &\int_0^{\hat{T}} \partial_y u(s, Y(kC)(s)) \{Y(kC)(s) - Y(0)(s)\} \, ds \\ &\qquad = (\nabla U(kC), kC) \le U(kC) - U(0) < +\infty \end{aligned}$$

by the subgradient estimate. Thus, the claimed weak continuity follows from dominated convergence.

□

The preceding proposition shows that when $\eta > 0$ Hindy–Huang–Kreps preferences satisfy Assumption 1.1, except that strict monotonicity and concavity hold true only on $\{C \in \mathcal{M}_+ \mid \Delta C(\hat{T}) = 0\}$. This, however, will not impose any real problems for our further investigations concerning uniqueness, existence, and the characterization of optimal consumption plans.

The whole analysis also goes through in case $\eta = 0$ when in addition the subgradient properties of Assumption 1.1 (i) and (ii) are partially violated; see Remarks 2.5 and 2.11.

Habit Formation is the notion that a high standard of living or level of satisfaction in the past increases the appetite for current consumption. To capture this effect together with the local substitutability of consumption over time, Hindy, Huang, and Zhu (1997) consider an extension of the above Hindy–Huang–Kreps utility involving a second average of past consumption $Z(C)$ which is interpreted as a habit formation index:

$$U(C) = \int_0^{\hat{T}} u(t, Y(C)(t), Z(C)(t))\, dt\,.$$

Hindy, Huang, and Zhu assume that the felicity function is increasing, concave, and differentiable in its 'space'–variables (y, z). With

$$Z(C)(t) \overset{\Delta}{=} \zeta e^{-\int_0^t \gamma(s)\, ds} + \int_0^t \gamma(s) e^{-\int_s^t \gamma(v)\, dv}\, dC(s) \quad (0 \leq t \leq \hat{T})\,,$$

the subgradient ∇U for this utility takes the form

$$\begin{aligned}\nabla U(C)(t) = &\int_t^{\hat{T}} \partial_y u(s, Y(C)(s), Z(C)(s)) \beta(t) e^{-\int_t^s \beta(u)\, du}\, ds \\ &+ \int_t^{\hat{T}} \partial_z u(s, Y(C)(s), Z(C)(s)) \gamma(t) e^{-\int_t^s \gamma(u)\, du}\, ds\,.\end{aligned}$$

Assumption 1.1 can be verified by the same methods as in the preceding section.

Chapter 2

The Utility Maximization Problem

This chapter is devoted to the study of the optimal consumption behavior which is induced by the preferences specified in the preceding chapter. To this end, we consider an economic agent whose preferences are described by some utility functional satisfying Assumption 1.1. We assume that he can invest some given initial wealth as a price–taker in a financial market. The problem is then to find the agent's most preferred consumption plan in his budget–feasible set.

As a first step, we give a general existence and uniqueness result for this optimization problem by adapting techniques from Cuoco (1997). The corresponding Theorem 2.1 covers both complete and incomplete financial markets, and it even allows for convex constraints on the agent's optimal portfolio strategy. Uniqueness of a solution immediately follows from strict concavity of the agent's preferences. The existence–proof is based on two building blocks: the duality between attainable contingent claims and the set of $\mathbb{P}$–equivalent risk–neutral measures on the one hand, and Komlós' compactness principle for L^1–bounded sequences of random variables on the other hand. The duality theorem — see Föllmer and Kramkov (1997) for its most general formulation — identifies the agent's budget–feasible set of consumption patterns as a set of nonnegative, optional random measures which simultaneously satisfy a system of linear constraints. This identification yields closedness of the budget–feasible set in our consumption space $(\mathcal{C}, d_{\mathcal{C}})$. In its measure–valued version given by Kabanov (1999), Komlós' (1967) theorem entails a suitable compactness principle for the budget–feasible set. Under some natural assumptions which guarantee uniform integrability of the set of budget–feasible utilities, we finally obtain existence of an optimal plan by continuity of preferences on the budget–feasible set.

In a second step, we determine the structure of the optimal plan. For this purpose, two fundamentally different approaches have been proposed in the literature: the method of dynamic programming and the so–called martingale method. The dynamic programming approach consist in specifying a Markovian state dynamics, and in deriving and solving explicitly the Hamilton–Jacobi–Bellman (HJB) equation for the value function of the optimization problem. This technique was used by Merton (1971) for solving his original time–additive utility maximization problem. For Hindy–Huang–Kreps preferences, this program was carried out by Hindy, Huang, and Kreps (1992) in the case of certainty, by Hindy and Huang (1993) in a Brownian setting, and recently by Benth, Karlsen, and Reikvam (1999) in a setting with Lévy–processes. These authors identify the HJB–equation as an (integro–)partial differential equation in free boundary form. The free boundary separates the state–space into a region of consumption and a region of no consumption. This structure entails that, at least in a diffusion setting, optimal consumption plans are of singular form. The explicit construction of the free boundary, however, is typically a tedious task since it relies on a closed–form solution of the HJB–equation. Indeed, such a solution can usually only be derived under strong homogeneity assumptions on both the system dynamics (i.e. the financial market) and the target functional (i.e. the utility function). These assumptions are needed in order to reduce the HJB–equation essentially to an ordinary differential equation which, of course, is much easier to solve. This reveals the main disadvantages of the dynamic programming approach: first, it is a priori restricted to a Markovian setting and, second, it often needs for strong homogeneity assumptions in order to lead to an explicit solution.

For time–additive utility functionals it has been shown in, e.g., Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987) that the martingale method allows to overcome the aforementioned deficiencies of the dynamic programming approach. In fact, it allows for both a general semimartingale model of the financial market and inhomogeneous utility functionals. The main goal of this chapter is to extend the martingale method to the case of non–time–additive utility functionals such as those proposed by Hindy, Huang, and Kreps.

We restrict ourselves to the case of a complete financial market where the budget–restriction takes the particularly simple form of a single linear constraint. In this framework, Theorem 2.2 provides an infinite–dimensional version of the Kuhn–Tucker Theorem which characterizes the optimal consumption policy by necessary and sufficient first–order conditions. Similar to the time–additive case described in Cox and Huang (1989), these conditions have a well–known economic interpretation: at the optimum, the investor's marginal utility from consumption is always less than or equal to a constant multiple of the cost for consumption, and this condition is binding whenever consumption actually occurs.

For time–additive utility functionals satisfying the Inada–conditions, it is easy to find a consumption plan which meets such first–order conditions: simply choose the current consumption rate such that marginal utility from consumption always is the same fixed multiple of the state–price density. The preceding construction entails in particular, that the first–order condition is always binding. This, however, is in deep contrast to the singular form of optimal consumption patterns which has been established in the non–time–additive framework of Hindy–Huang–Kreps utilities, e.g., in Hindy and Huang (1993). In fact, singularity of the optimal plan with respect to Lebesgue–measure implies that the first–order condition is almost surely almost never binding! In particular, it typically is impossible to invert the first–order condition in order to solve for the optimal consumption plan. Hence, our infinite–dimensional Kuhn–Tucker Theorem merely yields a characterization, not a description of the optimal consumption policy. However, we can use this characterization to investigate the general structure of the solution — at least for the special case of Hindy–Huang–Kreps utilities.

The key concept coming out of our investigation is a stochastic process which we call the 'minimal level of satisfaction'. This process gives us a canonical lower bound for the investor's optimal level of satisfaction. Theorem 2.3 shows how this property allows us to reconstruct the optimal consumption plan explicitly in terms of the minimal level process. In addition, the minimal level process will be characterized by a new kind of stochastic representation problem which, therefore, serves as a substitute for the Hamilton–Jacobi–Bellman equation in our non–Markovian setup. The general study of such representation problems seems to be of independent mathematical interest and will thus be carried out separately in the next chapter.

2.1 Formulation of the Problem

Consider an economic agent with preferences given by an expected utility functional

$$V(C) = \mathbb{E}U(C)$$

where U satisfies our Assumption 1.1. Moreover, assume the agent has the opportunity to invest in a financial market which provides a money market account and (possibly) other more risky securities such as stocks or derivatives. Interest is paid at rate $r = (r(t),\ 0 \leq t \leq \hat{T})$, a bounded adapted process. For a given initial wealth $w \geq 0$, the agent's budget–feasible set of consumption plans is then given by

$$\mathcal{A}(w) \stackrel{\Delta}{=} \{C \in \mathcal{C} \mid \Psi(C) \leq w\}\,,$$

where $\Psi(C) \in [0, \infty]$ denotes the minimal initial capital needed to finance a given consumption plan $C \in \mathcal{C}$ by investing in the assets of the financial market. We assume this quantity can be expressed in the form

$$\Psi(C) \stackrel{\Delta}{=} \sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^* \int_0^{\hat{T}} e^{-\int_0^t r(s)\,ds}\, dC(t) \quad (C \in \mathcal{C}) \tag{2.1}$$

where $\mathcal{P}$ is a fixed nonempty set of $\mathbb{P}$–equivalent probability measures on $(\Omega, \mathcal{F})$. The specific choice of this set is determined by the risk–structure of the considered financial market.

Clearly, the investor's problem is to find the most preferred consumption plan in his budget–feasible set. Formally, this comes down to the problem to

$$\text{Maximize } V(C) = \mathbb{E}U(C) \text{ over all } C \in \mathcal{A}(w). \tag{2.2}$$

Remark 2.1 *Note that the above formulation allows for incomplete markets and, more generally, even for markets under convex constraints; see, e.g., Föllmer and Kabanov (1998), Föllmer and Kramkov (1997), Cvitanic and Karatzas (1992).*

As a special case, let us consider a model of a security market consisting of a riskless bond and a stock, and let us assume that the constraint consists in excluding short selling of the stock. Föllmer and Kramkov (1997) show that this economic setting is captured by choosing

$$\mathcal{P} \triangleq \{\mathbb{P}^* \sim \mathbb{P} \,|\, \mathbb{P}^* \text{ is a supermartingale measure for each } S \in \mathcal{S}\},$$

where $\mathcal{S}$ denotes the set of all gain processes which are attainable by some admissible strategy without short selling. More precisely, they prove that

$$\sup_{\mathbb{P}^* \in \mathcal{P}} \mathbb{E}^* \left[e^{-\int_0^{\hat{T}} r(s)\,ds} H \right]$$

is the minimal amount needed to hedge a given contingent claim $H \geq 0$ with maturity $\hat{T}$. Thus, the minimal budget which is needed to finance a consumption plan $C \in \mathcal{C}$ is given by formula (2.1).

2.2 Existence and Uniqueness

This section is devoted to the proof of existence and uniqueness of a solution for the utility maximization problem (2.2) under the following

Assumption 2.1 *The family of budget–feasible utilities $(U(C),\ C \in \mathcal{A}(w))$ is uniformly $\mathbb{P}$–integrable.*

This assumption is slightly stronger than the condition that problem (2.2) is well–posed because the latter assumption amounts to require merely $L^1(\mathbb{P})$–boundedness of the family $(U(C),\ C \in \mathcal{A}(w))$. In particular, Assumption 2.1 ensures that the value of problem (2.2) is finite and $\mathcal{A}(w) \subset \mathrm{Dom}(V)$.

A sufficient condition for Assumption 2.1 to be satisfied is given by the following

Lemma 2.2 *A product–measurable functional $U : \Omega \times \mathcal{M}_+ \to \mathbb{R}$ satisfies Assumption 2.1 if the following two conditions hold true:*

(i) *For some* $\alpha \in (0,1)$*, the power growth–condition*

$$|U(C)| \leq \text{const.} \left(1 + C(\hat{T})^{\alpha}\right) \quad \mathbb{P}\text{–}a.s. \text{ for all } C \in \mathcal{C} \tag{2.3}$$

is satisfied.

(ii) *There is a measure* $\hat{\mathbb{P}} \in \mathcal{P}$ *with density* $\hat{Z} \overset{\Delta}{=} \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}$ *satisfying*

$$\hat{Z}^{-1} \in L^{\hat{p}}(\mathbb{P}) \tag{2.4}$$

for some $\hat{p} > \frac{\alpha}{1-\alpha}$.

Proof : We show that $(U(C),\ C \in \mathcal{A}(w))$ is bounded in $L^p(\mathbb{P})$ where $p \overset{\Delta}{=} \frac{\hat{p}}{\alpha(1+\hat{p})} > 1$. Due to our growth condition (2.3), it suffices in fact to show this property for the family $(C(\hat{T})^{\alpha},\ C \in \mathcal{A}(w))$. For this, note that $\alpha p < 1$, and apply Hölder's inequality to get

$$\mathbb{E}\left[C(\hat{T})^{\alpha p}\right] \leq \mathbb{E}\left[C(\hat{T})\hat{Z}\right]^{\alpha p} \mathbb{E}\left[\hat{Z}^{-\frac{\alpha p}{1-\alpha p}}\right]^{1-\alpha p} \leq \text{const.}\, w^{\alpha p}\mathbb{E}\left[\hat{Z}^{-\hat{p}}\right]^{1-\alpha p}.$$

Note that, in connection with condition (2.4), this yields the desired $L^p(\mathbb{P})$–boundedness. The last estimate holds true since

$$\mathbb{E}\left[C(\hat{T})\hat{Z}\right] = \widehat{\mathbb{E}}\left[C(\hat{T})\right] \leq \text{const.}\, \widehat{\mathbb{E}}\left[\int_0^{\hat{T}} e^{-\int_0^t r(s)\,ds}\, dC(t)\right] \leq \text{const.}\, w$$

for all $C \in \mathcal{A}(w)$. □

Remark 2.3 (i) *Assumptions similar to those of Lemma 2.2 have been made for the case of time–additive functionals in Cox and Huang (1991) and Aumann and Perles (1965). The example in Kramkov and Schachermayer (1999) suggests that a growth condition like* (2.3) *may in fact be necessary. An integrability condition similar to* (2.4) *can be found in Cuoco (1997).*

(ii) *For the Hindy–Huang–Kreps utilities of Section 1.2.2, growth condition* (2.3) *holds true, e.g., if for some* $\alpha \in (0,1)$*, the felicity function satisfies* $|u(t,y)| \leq \text{const.}\,(1+y^{\alpha})$ *for all* $y > 0$ *uniformly in* $t \in [0,\hat{T}]$.

The following is the main result of this section.

Theorem 2.1 *Under Assumptions 1.1 and 2.1, the utility maximization problem* (2.2) *has a unique solution.*

Proof : Uniqueness of an optimal consumption plan follows as usual from strict concavity by considering a convex combination of two optimal plans.

To prove existence, choose a maximizing sequence $C^n \in \mathcal{A}(w)$ $(n = 1, 2, \ldots)$ for (2.2). As the interest rate process r is bounded, we have

$$\sup_{C \in \mathcal{A}(w)} \mathbb{E}^* C(\hat{T}) < +\infty$$

for any $\mathbb{P}^* \in \mathcal{P}$. Hence, the sequence $(C^n(\hat{T}),\ n = 1, 2, \ldots)$ is bounded in $L^1(\mathbb{P}^*)$, and we may thus use Kabanov's version of Komlós' Theorem (Kabanov (1999), Lemma 3.5; Komlós (1967)), to obtain existence of a subsequence, again denoted by (C^n), which is almost surely weakly Cesaro convergent to some $C^* \in \mathcal{C}$, i.e., almost surely we have

$$\tilde{C}^n(t) \triangleq \frac{1}{n} \sum_{k=1}^{n} C^k(t) \to C^*(t) \quad (n \uparrow +\infty)$$

for $t = \hat{T}$ and also for every point of continuity t of C^*. In particular, we have $d_{\mathcal{C}}(\tilde{C}^n, C^*) \to 0$.

We claim that C^* is optimal for (2.2). Indeed, since $\gamma(t) \triangleq \exp\left(-\int_0^t r(s)\, ds\right)$ is continuous in t, we have

$$\int_0^{\hat{T}} \gamma(t)\, dC^*(t) = \lim_n \int_0^{\hat{T}} \gamma(t)\, d\tilde{C}^n(t) \quad \mathbb{P}\text{–a.s.}$$

Hence, by Fatou's lemma,

$$\mathbb{E}^* \int_0^{\hat{T}} \gamma(t)\, dC^*(t) \le \liminf_n \mathbb{E}^* \int_0^{\hat{T}} \gamma(t)\, d\tilde{C}^n(t) \le w,$$

for every $\mathbb{P}^* \in \mathcal{P}$, i.e., $C^* \in \mathcal{A}(w)$. Furthermore, like (C^n), also $(\tilde{C}^n)$ is a maximizing sequence for (2.2) by concavity of V. By Assumption 2.1, $(U(\tilde{C}^n),\ n = 1, 2, \ldots)$ is uniformly $\mathbb{P}$–integrable such that convergence of $\tilde{C}^n \to C^*$ in the metric $d_{\mathcal{C}}$ allows us to use continuity of preferences (Proposition 1.4 (iii)) to obtain $V(\tilde{C}^n) \to V(C^*)$ for $n \uparrow +\infty$. This proves optimality of C^* in $\mathcal{A}(w)$. $\square$

Remark 2.4 *The above argument also extends to the infinite horizon case $\hat{T} = +\infty$ provided U is almost surely continuous with respect to the vague topology of nonnegative σ–finite Borel–measures on $[0, +\infty)$.*

PROOF : As in the preceding proof, let $C^n \in \mathcal{A}(w)$ $(n = 1, 2, \ldots)$ be a maximizing sequence for V. By Kabanov's version of Komlós L^1–compactness principle, there is a subsequence $(n_k^1,\ k = 1, 2, \ldots)$ such that $(dC^{n_k^1}|_{[0,1]},\ k = 1, 2, \ldots)$ and all its subsequences are almost surely weakly Cesaro convergent to some Borel–measure $dC^{(1)}$ on $[0, 1]$. For $T = 1, 2, \ldots$ we pass to further subsequences $(n_k^{T+1},\ k = 1, 2, \ldots)$ if necessary to ensure in addition almost sure weak Cesaro–convergence of $(dC^{n_k^{T+1}}|_{[0,T]}$ to some measure $dC^{(T)}$ on $[0, T]$.

By construction, the nonnegative finite Borel–measures $dC^{(T)}$ $(T = 1, 2, \ldots)$ are consistent in the sense that $dC^{(T+1)}|_{[0,T]} = dC^{(T)}$ for all $T = 1, 2, \ldots$. They thus extend to a unique σ–finite Borel–measure dC^* on $[0, +\infty)$. This measure is almost surely the vague Cesaro–limit of the diagonal sequence $dC^{n_k^k}$ $(k = 1, 2, \ldots)$, i.e., the distribution functions almost surely converge in the sense that

$$\tilde{C}^m(t) \overset{\Delta}{=} \frac{1}{m} \sum_{k=1}^{m} C^{n_k^k}(t) \to C^*(t) \quad (m \uparrow +\infty)$$

for $t = 1, 2, \ldots$ and for every point of continuity t of $C^* = \int_0^{\cdot} dC^*$.

We claim that C^* is the optimal policy for an infinite time horizon. Indeed, as before, concavity and continuity (now with respect to vague convergence) of U ensure $V(C^*) = \sup_{C \in \mathcal{A}(w)} V(C)$. In addition, we have for $T = 1, 2, \ldots$ that

$$\int_0^T \gamma(t)\, dC^*(t) = \lim_m \int_0^T \gamma(t)\, d\tilde{C}^m(t)$$

almost surely, whence

$$\mathbb{E}^* \int_0^T \gamma(t)\, dC^*(t) \leq \liminf \mathbb{E}^* \int_0^T \gamma(t)\, d\tilde{C}^m(t) \leq w\,.$$

Letting $T \uparrow +\infty$, this proves $C^* \in \mathcal{A}(w)$ by monotone convergence. Hence, C^* is the optimal budget–feasible plan. $\square$

Remark 2.5 *Recall that for Hindy–Huang–Kreps utilities strict concavity holds true only on the slightly smaller set of consumption patterns*

$$\mathcal{C}' \overset{\Delta}{=} \{C \in \mathcal{C} \mid \Delta C(\hat{T}) = 0\}\,,$$

and, thus, Theorem 2.1 cannot be applied directly in this case.

Nevertheless we have existence and uniqueness of optimal plans as stated in the previous theorem also for such preferences. Indeed, the existence part of the above proof only uses concavity, not strict concavity. Moreover, the proof of uniqueness uses strict concavity only on the set of optimal *plans. For the case of Hindy–Huang–Kreps utilities,* *these plans are in fact contained in the above set $\mathcal{C}'$ because taking a gulp at the terminal date $\hat{T}$ does not effect the agent's utility and is thus suboptimal.*

Also for the infinite time horizon case, we have existence of a unique optimal policy for the Hindy–Huang–Kreps utility maximization problem, provided the felicity function u is negative and satisfies Assumption 1.2. This follows from Remark 2.4.

Indeed, Hindy–Huang–Kreps utilities U based on such a felicity are continuous in the vague topology, as the level of satisfaction at each fixed time t depends only on $C|_{[0,t]}$ and $(u(.,Y(C)(.)),\ C \in \mathcal{M}_+)$ is uniformly Lebesgue–integrable by assumption on u.

2.3 Solutions in the Complete Case

From now on we work under Assumption 2.1. In addition we make

Assumption 2.2 *The financial market is complete in the sense that $\mathcal{P}$ is a singleton.*

Thus, there is precisely one risk neutral measure $\mathbb{P}^*$ and we let ψ denote the RCLL–version of its associated state–price density

$$\psi(t) \stackrel{\Delta}{=} e^{-\int_0^t r(s)\,ds} \left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} \qquad (0 \le t \le \hat{T})\,.$$

Remark 2.6 *Clearly, the assumption of a complete financial market is very restrictive. For the case of utility from terminal wealth and for classical time–additive utilities, however, the optimal consumption policy in an incomplete market coincides with the optimal policy in some associated auxiliary complete market under appropriate assumptions; see, e.g., Cvitanic and Karatzas (1992) and Kramkov and Schachermayer (1999). This suggest a similar approach in our context, i.e., one could address the incomplete problem by first determining an appropriate auxiliary complete market and using then the methods developed in the remaining part of this chapter to describe the optimal policy.*

2.3.1 First Order Conditions for Optimality

The assumption of a complete financial market reduces the investor's optimization problem (2.2) to the problem of maximizing a concave functional under a *single* linear constraint. In finite dimensions, solutions to problems of this type are characterized by the well–known Kuhn–Tucker Theorem. It provides necessary and sufficient first–order conditions for an optimum in terms of gradients and Lagrange multipliers. The main aim of this section is to establish an analogue of this result in our infinite–dimensional context.

Theorem 2.2 *Under Assumptions 1.1, 2.1 and 2.2, a consumption plan $C^* \in \mathcal{C}$ solves the utility maximization problem* (2.2) *if and only if the following conditions (i)–(iii) hold true for some finite Lagrange multiplier $M > 0$:*

(i) $\langle \psi, C^* \rangle = w$,

(ii) $\nabla V(C^*)(t) \leq M\psi(t)$ *for every* $t \in [0, \hat{T}]$ $\mathbb{P}$*–a.s.,*

(iii) $\langle \nabla V(C^*), C^* \rangle = \langle M\psi, C^* \rangle$*, i.e., for* $\mathbb{P}$*–a.e.* $\omega \in \Omega$*,* $C^*(\omega)$ *is flat off the set*

$$\{t \in [0, \hat{T}] \mid \nabla V(C^*)(\omega, t) = M\psi(\omega, t)\} .$$

PROOF : The sufficiency part follows from the very general Lemma 2.7 given below. Necessity follows from our subsequent Lemmata 2.9 and 2.10. The idea is to proceed along the same lines as in the proof of the finite–dimensional Kuhn–Tucker Theorem. Thus, in a first step, we shall show that the optimal policy C^* solves the problem linearized around this optimum (Lemma 2.9). In a second step, we characterize the solutions of such a linear problem (Lemma 2.10), and it follows that C^* has to satisfy the above conditions (i)–(iii). □

The following lemma establishes sufficiency of the first–order conditions in a very general setting.

Lemma 2.7 *Let $V : \mathcal{C} \to \overline{\mathbb{R}}$ be a functional with a nonnegative, optional subgradient $\nabla V(C) \geq 0$ at every point $C \in \mathcal{C}$ in the sense that*

$$V(C') \leq V(C) + \langle \nabla V(C), C' \rangle - \langle \nabla V(C), C \rangle \tag{2.5}$$

holds true for all C' for which the right–hand side makes sense in $\overline{\mathbb{R}}$.

Assume $C^* \in \mathcal{C}$ satisfies the first–order conditions

$$\nabla V(C^*) \leq M\psi \quad \textit{and} \quad \langle \nabla V(C^*), C^* \rangle = M \langle \psi, C^* \rangle$$

for some Lagrange multiplier $M \in (0, +\infty)$.

Then C^* is optimal in its class, i.e.,

$$V(C^*) = \max_{C \in \mathcal{C},\, \langle \psi, C \rangle \leq \langle \psi, C^* \rangle} V(C) ,$$

or we have

$$V(C) = -\infty \quad \textit{for all } C \in \mathcal{C} \textit{ such that} \quad \langle \psi, C \rangle < +\infty .$$

PROOF : Without loss of generality, we may assume that $V(C^*) < +\infty$ since otherwise the first alternative is trivially satisfied. Now, suppose the second alternative does not hold true, i.e., there is a $C^0 \in \mathcal{C}$ with $V(C^0) > -\infty$ and $\langle \psi, C^0 \rangle < +\infty$. As $\nabla V(C^*) \leq M\psi$, we then have $\langle \nabla V(C^*), C^0 \rangle < +\infty$ and, thus, we may apply the subgradient estimate with $C' \triangleq C^0$ and $C \triangleq C^*$ to infer

$$-\infty < V(C^0) \leq V(C^*) + \left\langle \nabla V(C^*), C^0 \right\rangle - \langle \nabla V(C^*), C^* \rangle \ .$$

This inequality yields $V(C^*) > -\infty$ and $w \triangleq \langle \nabla V(C^*), C^* \rangle / M < +\infty$. Now, in order to prove the claimed optimality of C^*, consider $C' \in \mathcal{C}$ with $\langle \psi, C' \rangle \leq w$ and $V(C') > -\infty$. Again, the first–order condition $\nabla V(C^*) \leq M\psi$ allows us to apply the subgradient estimate. This time it yields

$$V(C') \leq V(C^*) + \langle \nabla V(C^*), C' \rangle - \langle \nabla V(C^*), C^* \rangle$$

which is

$$\leq V(C^*) + M \langle \psi, C' \rangle - Mw \leq V(C^*) \, .$$

Thus C^* is indeed optimal. □

Remark 2.8 *In case that V does not take the value $+\infty$, the argument of the above proof shows that a plan C^* satisfying the first–order conditions can have an infinite price $\langle \psi, C^* \rangle = +\infty$ only if the maximization problem for finite budgets is ill–posed in the sense that $V(C) = -\infty$ for all $C \in \mathcal{C}$ with a finite price $\langle \psi, C \rangle < +\infty$.*

Lemmata 2.9 and 2.10 yield the necessity of the first–order conditions in Theorem 2.2.

Lemma 2.9 *Let $C^* \in \mathcal{A}(w)$ be optimal for* (2.2) *and let $\phi^* \triangleq \nabla V(C^*)$. Then C^* solves the linear problem*

$$\max_{C \in \mathcal{A}(w)} \langle \phi^*, C \rangle \ , \tag{2.6}$$

and the value of this problem is finite.

PROOF : Consider $C \in \mathcal{A}(w)$ and let $C^\varepsilon \triangleq \varepsilon C + (1-\varepsilon)C^*$ $(0 \leq \varepsilon \leq 1)$. By optimality of C^* and because of the subgradient property in Proposition 1.4 (ii), we have

$$0 \geq \frac{1}{\varepsilon}\{V(C^\varepsilon) - V(C^*)\} \geq \langle \nabla V(C^\varepsilon), C - C^* \rangle \tag{2.7}$$

Lower–semicontinuity of $\langle \nabla V(.), C \rangle$ (Proposition 1.4 (iii)) yields

$$\liminf_{\varepsilon \downarrow 0} \langle \nabla V(C^\varepsilon), C \rangle \geq \langle \nabla V(C^*), C \rangle \ . \tag{2.8}$$

We claim that

$$\lim_{\varepsilon \downarrow 0} \langle \nabla V(C^\varepsilon), C^* \rangle \quad \text{exists and is} \quad = \langle \nabla V(C^*), C^* \rangle \ . \tag{2.9}$$

In connection with (2.8), this claim allows us to deduce our assertion

$$\langle \nabla V(C^*), C^* \rangle \geq \langle \nabla V(C^*), C \rangle$$

by letting $\varepsilon \downarrow 0$ in (2.7).

Hence, it remains to prove (2.9). To this end, use $\nabla V = {}^o\nabla U$ to write

$$\langle \nabla V(C^\varepsilon), C^* \rangle = \mathbb{E}\left(\nabla U(C^\varepsilon), C^* \right) \ .$$

By Assumption 1.1 (iii), $(\nabla U(C^\varepsilon), C^*)$ almost surely tends to $(\nabla U(C^*), C^*)$ as $\varepsilon \downarrow 0$. We show below that this convergence holds true also in $L^1(\mathbb{P})$. This yields

$$\langle \nabla V(C^\varepsilon), C^* \rangle = \mathbb{E}\left(\nabla U(C^\varepsilon), C^* \right) \to \mathbb{E}\left(\nabla U(C^*), C^* \right) = \langle \nabla V(C^*), C^* \rangle \, ,$$

proving our assertion (2.9).

Let us finally prove the claimed $L^1(\mathbb{P})$–convergence $(\nabla U(C^\varepsilon), C^*) \to (\nabla U(C^*), C^*)$. Clearly, having already established almost sure convergence, it suffices to prove uniform $\mathbb{P}$–integrability of $((\nabla U(C^\varepsilon), C^*) \, , \ 0 \leq \varepsilon \leq 1/2)$. To this end, use $C^* \preceq C^\varepsilon/(1-\varepsilon)$ and the subgradient property of ∇U (Assumption 1.1 (ii)) to estimate

$$0 \leq (\nabla U(C^\varepsilon), C^*) \leq \frac{(\nabla U(C^\varepsilon), C^\varepsilon)}{1-\varepsilon} \leq \frac{U(C^\varepsilon) - U(0)}{1-\varepsilon} \, .$$

Due to Assumption 2.1, the latter quantity defines a uniformly $\mathbb{P}$–integrable family of random variables parameterized by $0 \leq \varepsilon \leq 1/2$. Therefore, the above estimate implies the desired uniform integrability of $((\nabla U(C^\varepsilon), C^*) \, , \ 0 \leq \varepsilon \leq 1/2)$. □

Let us now discuss the linear problem (2.6).

Lemma 2.10 *Let ϕ, ψ be two strictly positive, rightcontinuous and adapted processes. Then every solution C^* to the linear optimization problem*

$$\max_{C \in \mathcal{C}} \langle \phi, C \rangle \quad s.t. \quad \langle \psi, C \rangle \leq w \tag{2.10}$$

satisfies

$$\mathbb{E} \int_0^{\hat{T}} 1_{\{\phi(t) \neq M\psi(t)\}} \, dC^*(t) = 0 \, , \tag{2.11}$$

where

$$M \stackrel{\Delta}{=} \mathbb{P}\text{–}\operatorname{ess\,sup} \sup_{t \in [0,\hat{T}]} \frac{\phi(t)}{\psi(t)} \, .$$

PROOF :

1. We first show that the value v of the linear problem (2.10) is given by Mw. Indeed, it is easy to see that $v \leq Mw$. Moreover, for every $K < M$ the set

$$\left\{ \omega \in \Omega \;\middle|\; \sup_{t \in [0,\hat{T}]} \frac{\phi(\omega,t)}{\psi(\omega,t)} > K \right\}$$

has positive probability. Therefore, letting τ^K denote the stopping time

$$\tau^K \triangleq \inf \left\{ t \in [0,\hat{T}] \;\middle|\; \frac{\phi(t)}{\psi(t)} > K \right\}$$

we can find $c^K \geq 0$ such that $C^K \triangleq c^K 1_{[\tau^K,\hat{T}]} \in \mathcal{C}$ satisfies $\langle \psi, C^K \rangle = w$. We have

$$\begin{aligned} Mw \geq v \geq \langle \phi, C^K \rangle &= \mathbb{E}\left[c^K \phi(\tau^K) 1_{\{\tau^K < +\infty\}} \right] \\ &\geq \mathbb{E}\left[c^K K \psi(\tau^K) 1_{\{\tau^K < +\infty\}} \right] = K \langle \psi, C^K \rangle = Kw \,. \end{aligned}$$

Letting $K \uparrow M$ in the above inequality yields $v = Mw$.

2. Suppose that C^* is a solution to (2.10). Then by Step 1 and the definition of M

$$Mw = \langle \phi, C^* \rangle \leq M \langle \psi, C^* \rangle \leq Mw$$

implying (2.11).

□

Remark 2.11 *The sufficiency–argument for Theorem 2.2 and the arguments for the preceding two lemmata also work in case of Hindy–Huang–Kreps utilities satisfying Assumption 1.2. Consequently, the characterization of optimal consumption plans given by Theorem 2.2 is also true for such preferences.*

PROOF : Note first that the strict form of concavity and monotonicity of utility functions is never used in the above arguments. Moreover, for Hindy–Huang–Kreps utilities U, the weak lower–semicontinuity of $\nabla V(.) = {}^o\nabla U(.)$ needed for (2.8) follows from Fatou's lemma and Proposition 1.8 (vi). The same proposition also yields (2.9) and, thus, the proof of Lemma 2.9 remains valid. Lemma 2.10 does not involve any preferences at all. Hence, with these slight modifications, the proof of Theorem 2.2 also works for Hindy–Huang–Krep–utilities. □

2.3.2 The Structure of Optimal Consumption Plans

As in the finite–dimensional case, our infinite–dimensional version of the Kuhn–Tucker Theorem does not yield directly an explicit description of the optimum. However, we can use the characterization in Theorem 2.2 in order to obtain more information about the structure of the solution as we are going to show in this and the following section for the special case of Hindy–Huang–Kreps utilities. The main result of this analysis will be Theorem 2.3. This theorem provides an equation characterizing what we call the 'minimal level of satisfaction'. This is an adapted process $L = (L(t),\ 0 \leq t \leq \hat{T})$ which gives us a canonical lower bound for the investor's optimal level of satisfaction. As we shall see, this property allows us to construct the optimal consumption plan explicitly in terms of the minimal level process L. Thus, in our non–Markovian setup, the equation characterizing this level plays a similar role as the Hamilton–Jacobi–Bellman equation does in Dynamic Programming.

As a first application of Theorem 2.2, let us now prove a version of the Dynamic Programming Principle:

Proposition 2.12 *Let $S \leq \hat{T}$ be a stopping time. If $C^* \in \mathcal{C}$ is a solution to (2.2) then, $\mathbb{P}$–a.s., it also solves the problem*

$$\text{Maximize } V_S(C) \triangleq \mathbb{E}\left[U(C)\,|\,\mathcal{F}_S\right] \text{ subject to } C \equiv C^* \text{ on } [0,S) \text{ and } \Psi_S(C) \leq \Psi_S(C^*)$$

where

$$\Psi_S(C) \triangleq \frac{1}{\psi(S)}\mathbb{E}\left[\int_S^{\hat{T}} \psi(t)\,dC(t)\,\middle|\,\mathcal{F}_S\right] \quad (C \in \mathcal{C})$$

is the price–functional at time S. Thus, a consumption plan which is optimal at time zero is its best continuation at any time afterwards.

Proof : Using the first–order conditions satisfied by C^*, this can be shown by the same calculation as for the sufficiency part of Theorem 2.2, now carried out for the 'conditional bracket' $\langle .,.\rangle_S \triangleq \mathbb{E}\left[(.,.)\,|\,\mathcal{F}_S\right]$ instead of $\langle .,.\rangle = \mathbb{E}\,(.,.)$. □

For the remainder of this chapter, we are going to work under

Assumption 2.3 *The agent has Hindy–Huang–Kreps utility $U(.)$ with a felicity function u satisfying Assumption 1.2.*

Thus, we have

$$U(C) \triangleq \int_0^{\hat{T}} u(t, Y(C)(t))\,dt \quad (C \in \mathcal{C})$$

where

$$Y(C)(t) = \eta e^{-\int_0^t \beta(s)\,ds} + \int_0^t \beta(s) e^{-\int_s^t \beta(v)\,dv}\,dC(s)$$

denotes the investors level of satisfaction at time $t \in [0, \hat{T}]$. The function β is continuous and strictly positive, and we put $B(t) \triangleq \int_0^t \beta(s)\, ds$. The constant $\eta \geq 0$ describes the agent's initial level of satisfaction.

From Theorem 2.1 and Remark 2.5, we obtain existence and uniqueness of an optimal consumption plan for every choice of this initial level of satisfaction. In order to stress its dependence on this parameter, let us denote the associated optimal plan by $C^{M,\eta}$ where $M > 0$ is the Lagrange multiplier induced by our Kuhn–Tucker Theorem 2.2. The following lemma shows how the optimal plan $C^{M,\eta}$ depends on the initial level of satisfaction η:

Lemma 2.13 *Let $Y(.)$ and $\tilde{Y}(.)$ denote the functionals for the level of satisfaction with initial value η and $\tilde{\eta}$, respectively. Similarly, denote the associated Hindy–Huang–Kreps utility functionals by V and $\tilde{V}$, respectively. Suppose $0 \leq \eta \leq \tilde{\eta}$.*

Then the respective optimal levels of satisfaction $Y^* \triangleq Y(C^{M,\eta})$, $\tilde{Y}^* \triangleq \tilde{Y}(C^{M,\tilde{\eta}})$ with the same *Lagrange multiplier $M > 0$ are related by*

$$\tilde{Y}^*(t) = \tilde{\eta} e^{-B(t)} \vee Y^*(t) \quad (0 \leq t \leq \hat{T})\,. \tag{2.12}$$

In particular, we have

$$dC^{M,\tilde{\eta}}(t) = 1_{\{\tau < t \leq \hat{T}\}}\, dC^{M,\eta}(t) + \tilde{\Delta}\, \delta_{\{\tau\}}(dt) \tag{2.13}$$

where the second summand is the Dirac measure with point mass

$$\tilde{\Delta} \triangleq \frac{1}{\beta(\tau)} \left(Y^*(\tau) - \tilde{\eta} e^{-B(\tau)} \right)$$

at time

$$\tau \triangleq \inf \left\{ t \geq 0 \;\middle|\; Y^*(t) \geq \tilde{\eta} e^{-B(t)} \right\}\,.$$

PROOF : Let $\tilde{C} \in \mathcal{C}$ be the consumption plan defined by the right side of (2.13). From the dynamics for the level of satisfaction (Remark 1.6), it may easily be deduced that $\tilde{Y}(\tilde{C})$ coincides with the right side of (2.12). Moreover, we see that $\tilde{Y}(\tilde{C}) = Y^*$ on $[\tau, \hat{T}]$. We will show that $\tilde{C}$ is optimal for the problem with initial level of satisfaction $\tilde{\eta}$ and that it has Lagrange multiplier $M > 0$. By uniqueness of this plan, we then obtain equations (2.12) and (2.13).

So let us verify the first–order conditions for optimality of $\tilde{C}$ with respect to $\tilde{V}$. For any stopping time $S \leq \hat{T}$, we have

$$\begin{aligned}
\nabla \tilde{V}(\tilde{C})(S) &= \mathbb{E}\left[\int_S^{\hat{T}} \partial_y u\left(t, \tilde{Y}(\tilde{C})(t)\right) \beta(S) e^{B(S)-B(t)}\, dt \,\middle|\, \mathcal{F}_S \right] \\
&\leq \mathbb{E}\left[\int_S^{\hat{T}} \partial_y u\left(t, Y^*(t)\right) \beta(S) e^{B(S)-B(t)}\, dt \,\middle|\, \mathcal{F}_S \right] = \nabla V(C^{M,\eta})(S) \qquad (2.14) \\
&\leq M\psi(S)\,, \qquad (2.15)
\end{aligned}$$

where inequality (2.14) follows from $\tilde{Y}(\tilde{C}) \geq Y^*$; inequality (2.15) is due to the first–order conditions satisfied by $C^{M,\eta}$. Since the above estimate holds true for any $S \in \mathcal{S}$, $\tilde{C}$ satisfies the first–order inequality constraint with Lagrange multiplier $M > 0$.

It remains to check the flat–off condition. Note first that $\operatorname{supp} d\tilde{C} \subset [\tau, \hat{T}]$. Moreover, we have $\tilde{Y}(\tilde{C}) = Y^*$ on $[\tau, \hat{T}]$ and, therefore, also $\nabla\tilde{V}(\tilde{C}) = \nabla V(C^{M,\eta})$ on this interval. Hence,

$$\begin{aligned}\left\langle \nabla\tilde{V}(\tilde{C}) - M\psi, \tilde{C} \right\rangle &= \mathbb{E}\int_\tau^{\hat{T}} \left\{\nabla\tilde{V}(\tilde{C})(t) - M\psi(t)\right\} d\tilde{C}(t) \\ &= \mathbb{E}\int_\tau^{\hat{T}} \left\{\nabla\tilde{V}(C^{M,\eta})(t) - M\psi(t)\right\} d\tilde{C}(t) = 0\end{aligned}$$

where the last equality is due to the absolute continuity of $d\tilde{C}$ with respect to $dC^{M,\eta}$ and to the flat–off condition satisfied by the latter consumption plan. □

Remark 2.14 *The preceding lemma shows in particular that it suffices to find the optimal consumption plan for $\eta = 0$. All other cases may be recovered from this one via equations* (2.12) *and* (2.13)*.*

We are now going to introduce the key concept of a 'minimal level of satisfaction'. Let us first motivate its definition by some heuristics.

For every stopping time $S < \hat{T}$, consider an agent, called S–Adam, who is born at time S. S–Adam starts with an initial level of satisfaction $\eta_S = 0$. Taking the history $\mathcal{F}_S$ as given, he solves

$$\text{Maximize } V_S(C) \triangleq \mathbb{E}\left[\int_S^{\hat{T}} u\left(t, Y_S(C)(t)\right)\, dt \,\middle|\, \mathcal{F}_S\right] \text{ subject to } \Psi_S(C) \leq w_S^M\ ,$$

where

$$Y_S(C)(t) \triangleq \int_S^t \beta(s) e^{-\int_s^t \beta(v)\,dv}\, dC(s) \quad (S \leq t \leq \hat{T})$$

denotes the evolution of S–Adam's level of satisfaction if, from his birth on, he follows the consumption plan C. We assume that, at his time of birth, S–Adam is endowed with the initial capital $w_S = w_S^M$ needed to buy the optimal consumption plan C_S^M which has Lagrange multiplier $M > 0$. This Lagrange multiplier M is also shared by all his brothers.

Now imagine that T–Adam, 'born' at the earlier stopping time $T \leq S$, thinks about his consumption from time S on. We claim that he can deduce his optimal behavior by observing his younger brother S–Adam. In fact, a (heuristic) application of the dynamic programming principle yields that S–Adam's and T–Adam's optimal consumption behavior should be related from time S on in the same way as the behavior of the two investors with differing initial levels of satisfaction that we considered in Lemma 2.13. This suggests that, as long as T–Adam's optimal level of satisfaction $Y_T(.) \stackrel{\Delta}{=} Y_T(C_T^M)(.)$ is strictly higher than S–Adam's, he should not consume, and that afterwards, when T–Adam's level of satisfaction has dropped to S–Adam's level, he should optimally mimic S–Adam's behavior. In particular, his optimal level of satisfaction at time S will be above S–Adam's level $Y_S(S)$. In fact, this holds true for all the elder 'brothers' of S–Adam.

Heuristically, we argue therefore that

$$L(S) = Y_S(S) \quad \text{for every stopping time} \quad S < \hat{T} \tag{2.16}$$

defines a *universal* lower bound from which we may recover *all* optimal consumption plans C_S^M $(S < \hat{T})$ with the same Lagrange multiplier $M > 0$. Indeed, every S–Adam should optimally consume 'just enough' to ensure that his level of satisfaction never falls below this lower bound. Lemma 2.15 below makes precise what we mean by 'consuming just enough' in this sense. We state this result only for time of birth being equal to zero, the general case can be treated analogously. Figures 4.1 and 4.2 in Section 4.2.3 below illustrate the way a consumption plan may be defined by this property.

Lemma 2.15 *Let $L = (L(t), 0 \leq t \leq \hat{T})$ be a real valued, progressively measurable process with upper–rightcontinuous paths. Set*

$$Y^L(0-) \stackrel{\Delta}{=} \eta, \quad Y^L(t) \stackrel{\Delta}{=} e^{-\int_0^t \beta(s)\,ds} \left(\eta \vee \sup_{0 \leq v \leq t} \left\{ L(v) e^{\int_0^v \beta(s)\,ds} \right\} \right) \quad (0 \leq t \leq \hat{T}).$$

(i) Y^L is an adapted RCLL–process of bounded variation with $Y^L \geq L$.

(ii) Consider the rightcontinuous process of bounded variation C^L defined by

$$C^L(0-) \stackrel{\Delta}{=} 0, \quad C^L(t) \stackrel{\Delta}{=} \int_0^t Y^L(s)\,ds + \int_0^t \beta(s)^{-1}\,dY^L(s) \quad (0 \leq t \leq \hat{T}).$$

This process is nondecreasing and adapted, and defines, therefore, a consumption plan, i.e., $C^L \in \mathcal{C}$.

(iii) The level of satisfaction induced by C^L, $Y(C^L)$, coincides with Y^L and is minimal above L in the following sense:

$$Y(C^L)(t) = Y^L(t) = \min_{C \in \mathcal{C},\, Y(C) \geq L} Y(C)(t) \quad \textit{for all} \quad 0 \leq t \leq \hat{T}.$$

In addition, if, for fixed $\omega \in \Omega$, time $t \in [0, \hat{T}]$ is a point of increase of $C^L(\omega, .)$ then $Y(C^L)(\omega, t) = L(\omega, t)$.

Definition 2.16 *We say, an investor following the plan C^L of the preceding lemma consumes just enough to keep his level of satisfaction always above L. Equivalently, we will say that the consumption plan C^L tracks the level process L.*

PROOF : Consider a consumption plan $C \in \mathcal{C}$. By definition of $Y(C)$, the process $A(C)$ defined by

$$A(C)(0-) \stackrel{\Delta}{=} \eta, \qquad A(C)(t) \stackrel{\Delta}{=} e^{B(t)} Y(C)(t) \quad (0 \leq t \leq \hat{T})$$

is increasing and adapted. In terms of $A(C)$, the restriction $Y(C) \geq L$ may be rewritten as

$$A(C)(t) \geq e^{B(t)} L(t) \quad \text{for all} \quad 0 \leq t \leq \hat{T} \,.$$

Obviously, the minimal increasing process A^L which starts in $A^L(0-) \stackrel{\Delta}{=} \eta$ and dominates the right side of this inequality is the running supremum

$$A^L(t) \stackrel{\Delta}{=} \sup_{0 \leq v \leq t} \{\eta \vee e^{B(v)} L(v)\} \quad (0 \leq t \leq \hat{T}) \,.$$

As L is progressively measurable, we may deduce from Théorème IV.2.33 in Dellacherie and Meyer (1975) that A^L is progressively measurable, too. Due to the upper–rightcontinuity of L, A^L even is an adapted RCLL–process. Of course, this also holds true for $Y^L(t) = e^{-B(t)} A^L(t)$ $(t \geq 0)$. In addition, we obtain that

$$dC^L(t) = \frac{1}{\beta(t)} e^{-B(t)} \, dA^L(t) \quad (0 \leq t \leq \hat{T})$$

defines a nonnegative, finite, and optional random measure with $Y(C^L) = Y^L$. Minimality of Y^L is inherited from the minimality of A^L. Finally, t is a point of increase of $C^L(\omega, .)$ iff it is a point of increase of $A^L(\omega, .)$. By upper–rightcontinuity of L, the latter implies $A^L(\omega, t) = e^{B(t)} L(\omega, t)$ which is equivalent to $Y(C^L)(\omega, t) = L(\omega, t)$. □

2.3.3 The Minimal Level Equation

The heuristic arguments of the preceding section suggest that, for a given Lagrange multiplier $M > 0$, there exists a canonical lower bound $L = L^M$ for the investor's level of satisfaction from which the optimal consumption behavior may be recovered as described in Lemma 2.15. However, the heuristic approach (2.16) to construct this minimal level sketched above is far from being constructive. Therefore, we would like to derive additional properties of this process that allow us to characterize it more explicitly.

To this end, let us continue our heuristics and recall that by assumption the felicity function u satisfies the Inada–condition

$$\partial_y u(t, 0+) = +\infty \quad \text{for all} \quad t \in [0, \hat{T}] \,.$$

Through our Kuhn–Tucker conditions, this implies that every S–Adam immediately starts consuming at his time of birth S. Indeed, otherwise his optimal level of satisfaction

$Y_S(.) = Y_S(C_S^M)(.)$ would remain zero over an open time interval, contradicting the inequality restriction

$$\nabla V_S(C_S^M)(s) \triangleq \mathbb{E}\left[\int_s^{\hat{T}} \partial_y u\left(t, Y_S(C_S^M)(t)\right)\beta(s)e^{B(s)-B(t)}\,dt\,\middle|\,\mathcal{F}_s\right] \leq M\psi(s) \quad (S \leq s \leq \hat{T})$$

for optimal plans. Hence, at time $s = S$, the first–order condition is binding for S–Adam and, therefore, we obtain the following equality:

$$\nabla V_S(C_S^M)(S) = \mathbb{E}\left[\int_S^{\hat{T}} \partial_y u(t, Y_S(t))\beta(S)e^{B(S)-B(t)}\,dt\,\middle|\,\mathcal{F}_S\right] = M\psi(S)\,. \tag{2.17}$$

As pointed out above, we conjecture that S–Adam's optimal consumption plan tracks some level process L. Given that his is indeed the case, Lemma 2.15 (adapted for initial time S and initial satisfaction zero) allows us to rewrite equation (2.17) in terms of this process L:

$$\mathbb{E}\left[\int_S^{\hat{T}} \partial_y u\left(t, e^{-B(t)} \sup_{S\leq v\leq t}\left\{L(v)e^{B(v)}\right\}\right)\beta(S)e^{B(S)-B(t)}\,dt\,\middle|\,\mathcal{F}_S\right] = M\psi(S)\,.$$

Since L is a universal lower bound for *every* S–Adam's level of satisfaction, this equality should hold true for *every* stopping time $S < \hat{T}$. In fact, together with the preceding heuristics and the following assumption it justifies the formal Definition 2.17 of the minimal level of satisfaction given below.

Assumption 2.4 *For every $M > 0$, there is a unique progressively measurable process $L = L^M \geq 0$ with upper–rightcontinuous paths and $L(\hat{T}) = 0$ such that the 'minimal level equation'*

$$\mathbb{E}\left[\int_S^{\hat{T}} \partial_y u\left(t, \sup_{S\leq v\leq t}\left\{L(v)e^{B(v)-B(t)}\right\}\right)\beta(S)e^{B(S)-B(t)}\,dt\,\middle|\,\mathcal{F}_S\right] = M\psi(S)\,. \tag{2.18}$$

is satisfied on $\{S < \hat{T}\}$ for every stopping time S.

We will carry out a general analysis of equations like (2.18) in the next chapters, to which we also defer the discussion of Assumption 2.4; see Corollary 4.1 for the case of certainty and Corollary 4.11 for the stochastic framework. For the moment, we just note that the above assumption allows us to give the following

Definition 2.17 *The process $L = L^M$ of Assumption 2.4 which is associated with $M >$ 0 will be called the minimal level of satisfaction for Lagrange multiplier M.*

The following theorem establishes the usefulness of this concept.

Theorem 2.3 *Under Assumptions 2.2–2.4, the consumption plan C^L which tracks the minimal level of satisfaction $L = L^M$ is optimal for the utility maximization problem (2.2); the constant $M > 0$ is its associated Lagrange multiplier.*

Proof : As Hindy–Huang–Kreps utilities have in particular the general subgradient property (2.5) needed for our Sufficiency Lemma 2.7, it suffices to verify that C^L satisfies the first–order conditions

$$\nabla V(C^L) \le M\psi \quad \text{and} \quad \left\langle \nabla V(C^L), C^L \right\rangle = \left\langle M\psi, C^L \right\rangle .$$

To this end, use the definition of C^L to write the induced level of satisfaction at time t as

$$Y(C^L)(t) = \left(Y(C^L)(s)e^{B(s)-B(t)}\right) \vee \sup_{s \le v \le t} \left\{L(v)e^{B(v)-B(t)}\right\}$$

for all $0 \le s \le t \le \hat{T}$. This shows that, for any $C \in \mathcal{C}$, we have

$$\begin{aligned}
\left\langle \nabla V(C^L), C \right\rangle &= \mathbb{E}(\nabla U(C^L), C) \\
&= \mathbb{E}\int_0^{\hat{T}} \left\{ \int_s^{\hat{T}} \partial_y u \left(t, \left\{Y(C^L)(s)e^{B(s)-B(t)}\right\} \vee \sup_{s \le v \le t} \left\{L(v)e^{B(v)-B(t)}\right\} \right) \right. \\
&\qquad\qquad \left. \cdot \; \beta(s)e^{B(s)-B(t)} \, dt \right\} dC(s) \\
&\le \mathbb{E}\int_0^{\hat{T}} \left\{ \int_s^{\hat{T}} \partial_y u \left(t, \sup_{s \le v \le t} \left\{L(v)e^{B(v)-B(t)}\right\} \right) \beta(s)e^{B(s)-B(t)} \, dt \right\} dC(s) .
\end{aligned} \tag{2.19}$$

By formula (1.1), we may replace the $\{\dots\}$–term in the last expression by its optional projection which, due to our minimal level equation (2.18), is given by $M\psi 1_{[0,\hat{T})}$. This yields

$$\left\langle \nabla V(C^L), C \right\rangle \le \mathbb{E}\int_0^{\hat{T}} M\psi(s) 1_{[0,\hat{T})}(s) \, dC(s) \le \langle M\psi, C \rangle . \tag{2.20}$$

As this estimate holds true for any $C \in \mathcal{C}$, we obtain the inequality condition $\nabla V(C^L) \le M\psi$ via Meyer's optional section theorem. To prove the flat–off condition, recall from Lemma 2.15 that, for any $\omega \in \Omega$, the measure $dC^L(\omega, .)$ charges only the set $\{s \in [0, \hat{T}] \mid Y(C^L(\omega))(s) = L(\omega, s)\}$. Thus, for $C = C^L$, estimate (2.19) is tight. Moreover, $\Delta C^L(\hat{T}) = 0$ because L is nonnegative with $L(\hat{T}) = 0$ by assumption. Consequently we find that for $C = C^L$ equality holds everywhere in (2.20). This shows

$$\left\langle \nabla V(C^L), C^L \right\rangle = \left\langle M\psi, C^L \right\rangle$$

which is the desired flat–off condition. □

The preceding theorem suggests the following method to construct explicit solutions to the Hindy–Huang–Kreps utility maximization problem in a complete financial market:

1. For every $M > 0$, find the progressively measurable process $L = L^M$ which solves the minimal level equation (2.18).

2. For each $M > 0$, compute the price $\Psi(C^M)$ of the consumption plan $C^M \triangleq C^{L^M}$ which tracks the minimal level of satisfaction L^M.

3. The consumption plan $C^{M(w)}$ with $\Psi(C^{M(w)}) = w$ is then the unique solution to the investor's utility maximization problem (2.2).

In Chapter 4, we will carry out this program and derive explicit solutions under appropriate conditions. Beforehand, we investigate equations like our minimal level equation (2.18) in the following Chapter 3.

Chapter 3

A Stochastic Representation Problem

In this chapter, we address a new kind of representation problem for optional processes which is inspired by our minimal level equation (2.18) of the previous chapter. More precisely, we consider a function $f = f(t, l)$, jointly continuous in $(t, l) \in [0, \hat{T}] \times \mathbb{R}$ and strictly decreasing in l, and ask whether a given optional process X can be written as an optional projection of the form

$$X = {}^{O}\left(\int_s^{\hat{T}} f(t, \sup_{s \le v \le t} L(v))\, dt,\ 0 \le s \le \hat{T} \right)$$

for some progressively measurable process L.

We start with a general uniqueness result and show in Theorem 3.1 that, up to optional sections, there can be at most one upper–rightcontinuous, progressively measurable solution L to the above representation problem.

For the question of existence, we first focus on the case when X is given by a deterministic function $x : [0, \hat{T}] \to \mathbb{R}$, i.e., we look for a deterministic function l such that

$$x(s) = \int_s^{\hat{T}} f(t, \sup_{s \le v \le t} l(v))\, dt \quad \text{for all} \quad 0 \le s \le \hat{T}\,.$$

Our construction of such a function l is based on an inhomogeneous notion of convexity which allows us to account for the time–inhomogeneity introduced by the function f. We develop analogues to the basic properties of usual convexity. In particular, we introduce the inhomogeneously convex envelope of a given function. In terms of these envelopes, we explicitly construct the solution l to the above problem if x is lower–semicontinuous. More precisely, Theorem 3.2 and its converse Theorem 3.3 reveal that precisely the lower–semicontinuous functions x with $x(\hat{T}) = 0$ can be represented in the above form when l varies over the deterministic upper–semicontinuous functions.

Existence of a solution in the general stochastic case is established in Theorem 3.4. The proof of this theorem uses techniques developed by El Karoui and Karatzas (1994) in their investigation of Gittins' problem of optimal dynamic scheduling. The main idea is to consider a family of auxiliary optimal stopping problems of Gittins–type whose value functions in the end allow us to describe the solution to our original representation problem. These auxiliary Gittins–problems are analyzed by means of the 'théorie generale' of Snell–envelopes as it is developed in El Karoui (1981). The exposition of this part owes very much to personal communication with Nicole El Karoui who suggested the convenient approach via the Gittins–index and the Envelope Theorem.

3.1 General Formulation

Let $X = (X(t),\ 0 \le t \le \hat{T})$ be an optional process on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions. Assume $X(\hat{T}) = 0$ and $X(S) \in L^1(\mathbb{P})$ for every stopping time $S \le \hat{T}$. Consider furthermore a function f satisfying

Assumption 3.1 *The mapping $f : [0,\hat{T}] \times \mathbb{R} \to \mathbb{R}$ is continuous and, for any $t \in [0,\hat{T}]$, $f(t,.) : \mathbb{R} \to \mathbb{R}$ is strictly decreasing from $+\infty$ to $-\infty$.*

We ask under which conditions there is a progressively measurable process $L = (L(t),\ 0 \le t < \hat{T})$ such that X coincides with the optional projection

$$X = {}^{O}\left(\int_s^{\hat{T}} f(t, \sup_{s \le v \le t} L(v))\, dt,\ 0 \le s \le \hat{T} \right). \tag{3.1}$$

To which extent is such a process L uniquely determined?

Omitting the optimal projection, we can state this problem in the equivalent form of the stochastic backward equation

$$X(S) = \mathbb{E}\left[\int_S^{\hat{T}} f(t, \sup_{S \le v \le t} L(v))\, dt \,\middle|\, \mathcal{F}_S \right] \quad \mathbb{P}\text{–a.s. for all stopping times } S \le \hat{T}. \tag{3.2}$$

For ease of notation, let us introduce the following sets of stopping times:

$$\mathcal{S} \triangleq \{T : \Omega \to [0,\hat{T}] \mid T \text{ is a stopping time}\} \quad \text{and} \quad \hat{\mathcal{S}} \triangleq \{T \in \mathcal{S} \mid T < \hat{T}\ \mathbb{P}\text{–a.s.}\}.$$

Given a stopping time $S \in \mathcal{S}$, we shall furthermore make frequent use of

$$\mathcal{S}(S) \triangleq \{T \in \mathcal{S} \mid T \ge S\ \mathbb{P}\text{–a.s.}\} \quad \text{and} \quad \mathcal{S}^{>}(S) \triangleq \{T \in \mathcal{S} \mid T > S\ \mathbb{P}\text{–a.s. on } \{S < \hat{T}\}\}.$$

3.2 Uniqueness

As a first step to prove uniqueness of a solution to (3.1), let us note the following

Lemma 3.1 *If L is a progressively measurable process satisfying* (3.1)*, so is its upper–rightcontinuous modification*

$$\tilde{L}(t) \stackrel{\Delta}{=} \limsup_{s \searrow t} L(s) = \lim_{\varepsilon \downarrow 0} \sup_{s \in [t,(t+\varepsilon)\wedge \hat{T}]} L(s) \quad (0 \le t \le \hat{T})\,.$$

PROOF : Due to Théorème IV.2.33 in Dellacherie and Meyer (1975), the upper–rightcontinuous process $\tilde{L}$ is again progressively measurable. Moreover, we have for each $\omega \in \Omega$ and all $s \in [0, \hat{T})$ that

$$\sup_{s \le v \le t} L(\omega, v) = \sup_{s \le v \le t} \tilde{L}(\omega, v)$$

at every point $t \in (s, \hat{T})$ where the increasing function on the left side in this equation does not jump. Since, for fixed ω and s, this happens at most a countable number of times, we obtain

$$\int_s^{\hat{T}} f(t, \sup_{s \le v \le t} L(\omega, v))\, dt = \int_s^{\hat{T}} f(t, \sup_{s \le v \le t} \tilde{L}(\omega, v))\, dt$$

for every $s \in [0, \hat{T}]$ and all $\omega \in \Omega$. Consequently, we can indeed replace L by $\tilde{L}$ in (3.1) without changing the optional projection. □

By the preceding lemma, we may assume without loss of generality that a solution L to (3.1) has upper–rightcontinuous paths. Moreover, since the final value $L(\hat{T})$ does not play any role, we may equally assume $L(\hat{T}) \equiv -\infty$.

Before we give our uniqueness result, let us note that, for all $S \in \mathcal{S}$ and $T \in \mathcal{S}^{>}(S)$, there is an $\mathcal{F}_S$–measurable random variable $l_{S,T}$ with

$$\mathbb{E}\left[\, X(S) - X(T) \,|\, \mathcal{F}_S\right] = \mathbb{E}\left[\int_S^T f(t, l_{S,T})\, dt \,\middle|\, \mathcal{F}_S \right]. \tag{3.3}$$

By strict monotonicity of f (Assumption 3.1), this random variable is uniquely determined up to a $\mathbb{P}$–null set on $\{S < \hat{T}\}$. On the complement $\{S = \hat{T}\}$, we put $l_{S,T} \stackrel{\Delta}{=} -\infty$.

Theorem 3.1 *Under Assumption 3.1, any progressively measurable, upper–rightcontinuous solution L to our representation problem* (3.1) *satisfies*

$$L(S) = \mathbb{P}\text{–}\operatorname*{ess\,inf}_{T \in \mathcal{S}^{>}(S)} l_{S,T} \quad \textit{for every stopping time} \quad S \in \mathcal{S} \tag{3.4}$$

where $l_{S,T}$ is defined by (3.3)*.*

In particular, the solution to (3.1) *is uniquely determined on* $[0, \hat{T})$ *up to optional sections.*

PROOF : Fix a stopping time $S \le \hat{T}$. Consider $T \in \mathcal{S}^{>}(S)$ and use the representation property of L to write

$$X(S) = \mathbb{E}\left[\int_S^T f(t, \sup_{S \le v \le t} L(v))\, dt \,\middle|\, \mathcal{F}_S \right] + \mathbb{E}\left[\int_T^{\hat{T}} f(t, \sup_{S \le v \le t} L(v))\, dt \,\middle|\, \mathcal{F}_S \right].$$

As $f(t,.)$ is decreasing, we may estimate the first integrand from above by $f(t, L(S))$ and the second integrand by $f(t, \sup_{T \le v \le t} L(v))$ to obtain

$$X(S) \le \mathbb{E}\left[\int_S^T f(t, L(S))\, dt \,\middle|\, \mathcal{F}_S \right] + \mathbb{E}\left[\int_T^{\hat{T}} f(t, \sup_{T \le v \le t} L(v))\, dt \,\middle|\, \mathcal{F}_S \right].$$

From the representation property of L at time T, it follows that we may write the second of the above two summands as

$$\mathbb{E}\left[\int_T^{\hat{T}} f(t, \sup_{T \le v \le t} L(v))\, dt \,\middle|\, \mathcal{F}_S \right] = \mathbb{E}\left[X(T) \,|\, \mathcal{F}_S \right]$$

and, therefore, we get the estimate

$$\mathbb{E}\left[X(S) - X(T) \,|\, \mathcal{F}_S \right] \le \mathbb{E}\left[\int_S^T f(t, L(S))\, dt \,\middle|\, \mathcal{F}_S \right].$$

As $L(S)$ is $\mathcal{F}_S$–measurable (Dellacherie and Meyer (1975), Théorème IV.64b), this shows $L(S) \le l_{S,T}$ almost surely. Since in the above estimate $T \in \mathcal{S}^{>}(S)$ is arbitrary, we deduce

$$L(S) \le \mathbb{P}\text{–}\operatorname*{ess\,inf}_{T \in \mathcal{S}^{>}(S)} l_{S,T}.$$

For the converse inequality, consider the sequence of stopping times

$$T^n \triangleq \inf\{t \ge S \mid \sup_{S \le v \le t} L(v) \ge K^n\} \wedge \hat{T} \quad (n = 1, 2, \ldots)$$

where

$$K^n = (L(S) + 1/n) 1_{\{L(S) > -\infty\}} - n 1_{\{L(S) = -\infty\}}.$$

Observe that $T^n \in \mathcal{S}^{>}(S)$ due to the upper–rightcontinuity of L. Observe furthermore that this path property also implies

$$\sup_{S \le v \le t} L(v) = \sup_{T^n \le v \le t} L(v) \quad \text{for all} \quad t \in [T^n, \hat{T}).$$

Thus, we may write

$$\begin{aligned} X(S) &= \mathbb{E}\left[\int_S^{T^n} f(t, \sup_{S \le v \le t} L(v))\, dt \,\middle|\, \mathcal{F}_S \right] + \mathbb{E}\left[\int_{T^n}^{\hat{T}} f(t, \sup_{T^n \le v \le t} L(v))\, dt \,\middle|\, \mathcal{F}_S \right] \\ &\ge \mathbb{E}\left[\int_S^{T^n} f(t, K^n)\, dt \,\middle|\, \mathcal{F}_S \right] + \mathbb{E}\left[X(T^n) \,|\, \mathcal{F}_S \right], \end{aligned}$$

where the last estimate follows from our definition of T^n and from the representation property of L at time T^n. As K^n is $\mathcal{F}_S$–measurable, the above estimate allows us to deduce

$$K^n \geq l_{S,T^n} \geq \mathbb{P}\text{–} \operatorname*{ess\,inf}_{T \in \mathcal{S}^>(S)} l_{S,T} \,.$$

Now note that for $n \uparrow +\infty$, we have $K^n \downarrow L(S)$ and so we obtain

$$L(S) \geq \mathbb{P}\text{–} \operatorname*{ess\,inf}_{T \in \mathcal{S}^>(S)} l_{S,T} \,.$$

□

Remark 3.2 *(i) For the above uniqueness theorem we did not assume any path regularity for X explicitly. All we needed to establish uniqueness is the representation property* (3.2) *and upper–rightcontinuity of the progressive process L.*

(ii) In the special case where $f(t,l) = -\alpha e^{-\alpha t} l$ for some constant $\alpha > 0$, the characterization given by Theorem 3.1 takes the form

$$L(S) = \mathbb{P}\text{–} \operatorname*{ess\,inf}_{T \in \mathcal{S}^>(S)} \frac{\mathbb{E}\left[\, X(T) - X(S) \,|\, \mathcal{F}_S\right]}{\mathbb{E}\left[\, \int_S^T \alpha e^{-\alpha t}\, dt \,\middle|\, \mathcal{F}_S\right]} \quad (S \in \hat{\mathcal{S}}) \,. \tag{3.5}$$

Note that a remarkably similar representation has been derived for the Gittins–index in optimal dynamic scheduling by El Karoui and Karatzas (1994); see their equation (3.11). In fact, it was this similarity which motivated the approach taken in Section 3.4 to prove existence of a solution to our representation problem (3.1)*.*

Note furthermore that the above optimal stopping problem is not directly amenable to a solution following the standard approach via the Snell–envelope. For a discussion of optimal stopping problems similar to (3.5)*, we refer the reader also to Morimoto (1991).*

The above theorem characterizes any upper–rightcontinuous, progressively measurable solution to (3.1) up to optional sections. While this is not sufficient for indistinguishability of two such solutions, it is sufficient for indistinguishability of the induced running suprema as we show in the following

Lemma 3.3 *Let L, L' be two progressively measurable processes with upper–rightcontinuous paths. Assume $L(S) = L'(S)$ $\mathbb{P}$–a.s. for every stopping time S. Then the running supremum processes*

$$A(t) \triangleq \sup_{0 \leq v \leq t} L(v) \quad \textit{and} \quad A'(t) \triangleq \sup_{0 \leq v \leq t} L'(v) \qquad (t \geq 0)$$

are indistinguishable adapted processes with rightcontinuous paths.

PROOF : By Théorème IV.2.33 in Dellacherie and Meyer (1975), both A and A' are adapted processes. Moreover, upper–rightcontinuity of L and L' ensures that A and A' are rightcontinuous. Now, let $\varepsilon > 0$ and consider the stopping time

$$S \stackrel{\Delta}{=} \inf\{t \geq 0 \mid A(t) > A'(t) + \varepsilon\}\,.$$

We will show that

$$L(S) = A(S) \quad \text{on} \quad \{S < +\infty\}\,. \tag{3.6}$$

This yields

$$L(S) = A(S) \geq A'(S) + \varepsilon > L'(S) \quad \text{on} \quad \{S < +\infty\}$$

and, thus, $\{S < +\infty\}$ is contained in a $\mathbb{P}$–nullset, since $L(S) = L'(S)$ $\mathbb{P}$–a.s. by assumption. As this holds true for any $\varepsilon > 0$, we deduce $A \leq A'$ $\mathbb{P}$–a.s. Interchanging the roles of A and A' in the above argument yields the converse estimate $A \geq A'$ $\mathbb{P}$–a.s. and we are done.

So let us prove (3.6). Clearly,

$$L(S) \leq A(S) = A(S-) \vee L(S)$$

and it thus suffices to prove $A(S-) \leq L(S)$ on $\{S < +\infty\}$. For this we will give a pathwise argument and so we fix $\omega \in \{S < +\infty\}$. By definition, S is a point of increase of A, i.e., we have $A(\omega, S(\omega)-) < A(\omega, S(\omega) + \delta)$ for all $\delta > 0$. Thus, for every $\delta > 0$, there is some $t^\delta \in [S(\omega), S(\omega) + \delta]$ such that $A(\omega, S(\omega)-) < L(\omega, t^\delta)$. Now, let $\delta \downarrow 0$ to deduce from these inequalities that indeed

$$A(\omega, S(\omega)-) \leq \limsup_{\delta \downarrow 0} L(\omega, t^\delta) \leq L(\omega, S(\omega))$$

where the second estimate is due to the upper–rightcontinuity of $L(\omega, .)$. As this holds true for every $\omega \in \{S < +\infty\}$, we find $\{S < +\infty\} \subset \{A(S-) \leq L(S)\} = \{A(S) = L(S)\}$ which is precisely our claim (3.6). □

3.3 Existence in the Deterministic Case

It remains to check when the unique candidate (3.4) for a solution to our representation problem (3.1) does indeed solve this problem.

To this end, let us first study the case of certainty when X can be identified with some deterministic function $x : [0, \hat{T}] \to \mathbb{R}$ satisfying $x(\hat{T}) = 0$. In this case, Theorem 3.1 shows that the only candidate for an upper–rightcontinuous function $l : [0, \hat{T}) \to \mathbb{R}$ with

$$x(s) = \int_s^{\hat{T}} f(t, \sup_{s \leq v \leq t} l(v))\, dt \quad \text{for all} \quad 0 \leq s \leq \hat{T} \tag{3.7}$$

is characterized by

$$l(s) = \inf_{s<t\leq\hat{T}} l_{s,t} \tag{3.8}$$

where $l_{s,t} \in \mathbb{R}$ is the unique constant satisfying

$$x(s) - x(t) = \int_s^t f(u, l_{s,t})\, du\,.$$

As a motivation for our further steps to solve this problem, let us consider the following

Example 3.4 *Suppose $f(t,y) \equiv -y$. For this choice of f, it is easy to see that $l_{s,t}$ is the difference quotient*

$$l_{s,t} = \frac{x(t)-x(s)}{t-s} \quad (0 \leq s < t \leq \hat{T})$$

and, thus, $l(s)$ has to be the smallest slope of a secant in the graph of x which starts in $(s, x(s))$ and which ends in some point $(t, x(t))$ with $t > s$; compare Figure 3.1.

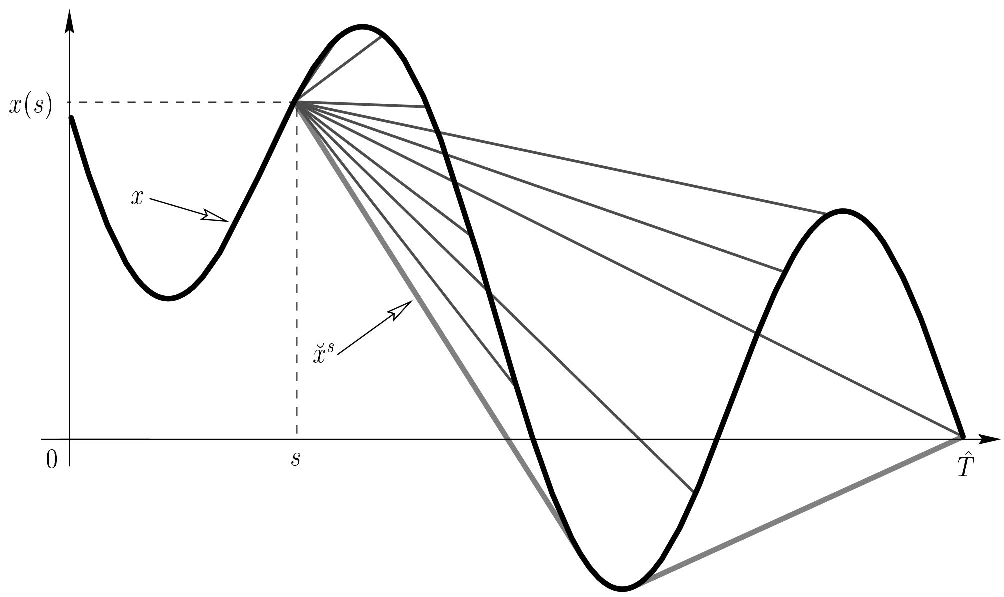

Figure 3.1: A function x (thick black line), its convex envelope $\breve{x}^s$ (thick grey line), and various secants (thin lines) starting in $(s, x(s))$.

Of course, if x is convex, this smallest slope is precisely the righthand derivative $\partial^+ x(s)$. The following calculation shows that for a continuous, convex function x

$$l(t) \stackrel{\Delta}{=} \partial^+ x(t) \quad (0 \leq t < \hat{T})$$

does in fact satisfy (3.7)*: Use* $x(\hat{T}) = 0$ *and the absolute continuity of* x *to write*

$$x(s) = x(s) - x(\hat{T}) = -\int_s^{\hat{T}} \partial^+ x(t)\, dt\,.$$

By convexity of x*, this is*

$$= -\int_s^{\hat{T}} \sup_{s \le v \le t} \partial^+ x(v)\, dt = \int_s^{\hat{T}} f(t, \sup_{s \le v \le t} \partial^+ x(v))\, dt$$

and we obtain the desired representation of x*.*

For general functions x*, the above figure suggests to consider the convex envelope* $\breve{x}^s$ *starting at time* s *and to use its initial slope* $(\partial^+ \breve{x}^s)(s)$ *in order to obtain the solution* l*.*

The above example shows that convexity may be a useful concept in the context of our deterministic representation problem (3.7). Indeed, as we shall see in the next sections, a suitably generalized notion of convexity allows us to construct the unique solution to (3.7) explicitly in terms of generalized convex envelopes.

3.3.1 Inhomogeneously Convex Functions

In this section we shall introduce an inhomogeneous notion of convexity which will prove to be useful for solving the deterministic representation problem (3.7). This special form of convexity accounts for the time–inhomogeneity introduced to our representation problem by the function f. As we shall see, it inherits many properties of usual convexity, the most important being a characterization in terms of derivatives and the existence of an inhomogeneously convex envelope of a given function.

As a framework for this section, we fix a compact interval $[a, b]$ on the real line, and we consider a continuous function $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ which is strictly *increasing* from $-\infty$ to $+\infty$ in its second argument.

Remark 3.5 *In our subsequent application to the representation problem* (3.7)*, the function* g *will be defined as* $g \triangleq -f$*.*

Now, let x be a real–valued function on $[a, b]$.

Definition 3.6 *We call* x *inhomogeneously convex with respect to* g*, or* g*–convex for short, if for all* $s, t, u \in [a, b]$ *such that* $s < t < u$ *we have*

$$x(t) \le x(s) + \int_s^t g(v, l_{s,u})\, dv \tag{3.9}$$

where $l_{s,u} \in \mathbb{R}$ *is the unique constant satisfying*

$$x(u) = x(s) + \int_s^u g(v, l_{s,u})\, dv\,. \tag{3.10}$$

We call x *strictly* g*–convex if we always have strict inequality in* (3.9).

Remark 3.7 *The preceding definition is equivalent to the usual definition of convexity in case the function* $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ *is time–homogeneous in the sense that it does not depend on its first argument.*

In complete analogy to usual convexity, there are the following alternative characterizations of g–convexity.

Proposition 3.8 *The following properties are equivalent:*

(i) x *is (strictly)* g*–convex.*

(ii) For all $s, t, u \in [a,b]$ *such that* $s < t < u$ *we have*

$$l_{s,t} \le l_{t,u} \qquad (\text{resp. } l_{s,t} < l_{t,u}) \tag{3.11}$$

where $l_{s,t}$ *and* $l_{t,u}$ *are defined as in* (3.10).

(iii) $x(a) \ge x(a+)$, $x(b) \ge x(b-)$, *and* x *is absolutely continuous on* (a,b) *with a density* $\dot{x}$ *of the form*

$$\dot{x}(t) = g(t, l(t)) \quad (t \in [a,b])$$

for some (strictly) increasing function $l : (a,b) \to \mathbb{R}$.

Proof : The argument for the characterization of strict convexity being similar, we only prove the characterization of convexity.

(i) ⇒ *(ii)* We shall show $l_{s,t} \le l_{s,u}$ and $l_{s,u} \le l_{t,u}$.

For the first inequality we note that, by definition of $l_{s,t}$ and (i),

$$\int_s^t g(v, l_{s,t})\, dv = x(t) - x(s) \le \int_s^t g(v, l_{s,u})\, dv\,.$$

Similarly, we obtain the second inequality from

$$\begin{aligned}\int_t^u g(v, l_{s,u})\, dv &= x(u) - \left(x(s) + \int_s^t g(v, l_{s,u})\, dv\right) \\ &\le x(u) - x(t) = \int_t^u g(v, l_{t,u})\, dv\,.\end{aligned}$$

$(ii) \Rightarrow (i)$ Using the definition of $l_{s,t}$ and $l_{t,u}$, we may write

$$x(u) - x(s) = \int_s^t g(v, l_{s,t})\, dv + \int_t^u g(v, l_{t,u})\, dv\,.$$

By (ii) and the definition of $l_{s,u}$, this yields

$$\int_s^u g(v, l_{s,u})\, dv \geq \int_s^u g(v, l_{s,t})\, dv\,.$$

Thus, $l_{s,t} \leq l_{s,u}$ and therefore

$$x(t) - x(s) = \int_s^t g(v, l_{s,t})\, dv \leq \int_s^t g(v, l_{s,u})\, dv$$

as was to be shown.

$(iii) \Rightarrow (ii)$ Because of the boundary conditions, it suffices to show (3.11) for $s, t, u \in (a, b)$. The monotonicity of $l(.)$ implies

$$x(t) - x(s) = \int_s^t g(v, l(v))\, dv \leq \int_s^t g(v, l(t))\, dv$$

which yields $l_{t,s} \leq l(t)$. Moreover,

$$x(u) - x(t) = \int_t^u g(v, l(v))\, dv \geq \int_t^u g(v, l(t))\, dv,$$

whence we deduce $l(t) \leq l_{t,u}$.

$(ii) \Rightarrow (iii)$ The same argument as in $(ii) \Rightarrow (i)$ shows that, for $t \in (a, b)$ fixed, both $l_{.,t}$ and $l_{t,.}$ are increasing functions on their respective domains. Hence, we may define

$$l^-(t) \triangleq \lim_{s \uparrow t} l_{s,t} \quad \text{and} \quad l^+(t) \triangleq \lim_{s \downarrow t} l_{t,s}\,.$$

By (3.11) we have, for $s < t < u$ in (a,b),

$$l_{s,t} \leq l^-(t) \leq l^+(t) \leq l_{t,u}\,.$$

In particular, both l^- and l^+ are increasing, real–valued functions on (a, b).

We next show that x is locally Lipschitz on (a, b). To this end, we fix $[a_0, b_0] \subset (a, b)$. For $s, t \in [a_0, b_0]$ with $s < t$, we have

$$|x(t) - x(s)| \leq \int_s^t |g(v, l_{s,t})|\, dv \leq \int_s^t |g(v, l^+_{b_0})| \vee |g(v, l^-_{a_0})|\, dv \leq k|t - s|$$

where k is a constant which depends on a_0, b_0 and g, but not on s or t. Hence, x is indeed Lipschitz continuous on any compact subinterval $[a_0, b_0] \subset (a, b)$.

As Lipschitz continuity implies absolute continuity, we may now deduce that the function x has a locally Lebesgue integrable density $\dot{x}$ on (a, b). In order to identify this density, recall that it is characterized by the almost everywhere existing limit

$$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \dot{x}(t) \,.$$

Now, let $t \in (a, b)$ be a point where the above limit exists. For $0 < h < \hat{T} - t$ we may write

$$\frac{x(t+h) - x(t)}{h} = \frac{1}{h} \int_t^{t+h} g(v, l_{t,t+h}) \, dv \,.$$

Since g is continuous and $l_{t,\cdot}$ is increasing, the right side of this equation converges to

$$g(t, \lim_{h \downarrow 0} l_{t,t+h}) = g(t, l^+(t))$$

as $h \downarrow 0$. Hence,

$$\lim_{h \downarrow 0} \frac{x(t+h) - x(t)}{h} = g(t, l^+(t))$$

Analogously, one can also show that for Lebesgue–a.e. $t \in (a, b)$

$$\lim_{h \uparrow 0} \frac{x(t+h) - x(t)}{h} = g(t, l^-(t)) \,.$$

In particular, we have

$$\dot{x}(t) = g(t, l^-(t)) = g(t, l^+(t))$$

for almost every $t \in (a, b)$. As both l^- and l^+ are increasing, either representation of $\dot{x}$ is of the desired form.

To check the boundary conditions $x(a) \geq x(a+)$, $x(b) \geq x(b-)$, we note first that the involved limits do exist. Indeed, fix $t_0 \in (a, b)$ and consider the function $\tilde{x}$ defined by

$$\tilde{x}(t) \triangleq x(t_0) + \int_{t_0}^t \left\{ g(s, l^+(s)) - g(s, l^+(t_0)) \right\} ds \quad (t \in (a, b)) \,.$$

Since l^+ is increasing, $\tilde{x}$ is increasing on (t_0, b) and decreasing on (a, t_0). In particular, it has limits for $t \downarrow a$ and $t \uparrow b$. Since by definition

$$\tilde{x}(t) = x(t) - \int_{t_0}^t g(s, l^+(t_0)) \, ds$$

this property carries over to x.

As we know already that property (ii) implies g–convexity, we have for any $t \in (a,b)$ the estimate

$$x(t) \leq x(a) + \int_a^t g(v, l_{a,b})\, dv = x(b) - \int_t^b g(v, l_{a,b})\, dv\,.$$

Obviously, the right side converges to $x(a)$ as $t \downarrow a$ and to $x(b)$ as $t \uparrow b$ while the left side converges to $x(a+)$ and $x(b-)$, respectively.

□

Like usual convexity, also g–convexity is stable with respect to taking suprema as we shall see in the proof of

Proposition 3.9 *For every function $x : [a,b] \to \mathbb{R}$ which is bounded from below, there exists a maximal g–convex function $\breve{x}$ which is dominated by x.*

This proposition allows us to give

Definition 3.10 *The function $\breve{x}$ of Proposition 3.9 will be called the g–convex envelope of x.*

Proof of Proposition 3.9 Since x is bounded from below, there are constants c_0, l_0 such that

$$c_0 + \int_a^t g(v, l_0)\, dv \leq x(t)$$

for all $t \in [a,b]$. In particular, there exists a g–convex function which is dominated by x and, therefore, the set $\mathcal{X}$ of these functions is not empty. We may thus consider the pointwise supremum

$$\breve{x}(t) \stackrel{\Delta}{=} \sup_{\xi \in \mathcal{X}} \xi(t) \quad (a \leq t \leq b)\,.$$

Clearly, $\breve{x} \leq x$ and we only need to show that $\breve{x}$ is g–convex. So fix $s < t < u$ in $[a,b]$ and consider $\xi \in \mathcal{X}$. Since ξ is g–convex, we have

$$\xi(t) \leq \Xi(\xi(s), \xi(u))$$

where

$$\Xi(\xi_1, \xi_2) \stackrel{\Delta}{=} \xi_1 + \int_s^t g(v, l)\, dv = \xi_2 - \int_t^u g(v, l)\, dv$$

with $l = l(\xi_1, \xi_2) \in \mathbb{R}$ such that

$$\xi_1 + \int_s^u g(v, l)\, dv = \xi_2\,.$$

It is easy to see that the function Ξ is increasing in both arguments. Thus,

$$\xi(t) \le \Xi(\breve{x}(s), \breve{x}(u))\,.$$

As this holds true for any $\xi \in \mathcal{X}$, we deduce

$$\breve{x}(t) \le \Xi(\breve{x}(s), \breve{x}(u)),$$

which means that indeed $\breve{x}$ is g–convex. □

Let us record some properties of g–convex envelopes in the following

Proposition 3.11 *Let $x : [a,b] \to \mathbb{R}$ be bounded from below and denote by*

$$x_*(t) \triangleq \liminf_{s \to t} x(s) \quad (a \le t \le b)$$

its lower–semicontinuous envelope. Then the g–convex envelope $\breve{x}$ has the following properties:

(i) $\breve{x}(a) = x(a)$, $\breve{x}(b) = x(b)$, and $\breve{x} \le x_*$ on (a,b).

(ii) Let $\breve{l}$ be the unique increasing, rightcontinuous function $(a,b) \to \mathbb{R}$ such that $g(.,\breve{l}(.))$ is a density for $\breve{x}$ on (a,b). Define

$$\breve{l}(a) \triangleq \lim_{t \downarrow a} \breve{l}(t) \quad \textit{and} \quad \breve{l}(b) \triangleq \lim_{t \uparrow b} \breve{l}(t)\,.$$

Then $\breve{l}$ induces a Borel–measure $d\breve{l}$ on (a,b) with

$$\operatorname{supp} d\breve{l} \subset \{\breve{x} = x_*\}\,.$$

(iii) If x satisfies $x(a) = x_*(a)$ and $x(b) = x_*(b)$, then $\breve{x}$ is absolutely continuous on $[a,b]$.

(iv) For $t \in [a,b)$, let $\breve{x}^t$ denote the g–convex envelope of the restriction $x|_{[t,b]}$. Then we have

$$(\partial^+ \breve{x}^{t_1})(s) \ge (\partial^+ \breve{x}^{t_2})(s)$$

for any $t_1 \le t_2 \le s$ in (a,b), and this inequality is strict iff

$$\breve{x}^{t_1}(s) < \breve{x}^{t_2}(s)\,.$$

PROOF :

(i) Let ξ be an arbitrary g–convex function dominated by x. Define $\tilde{\xi} \triangleq \xi$ on (a, b) and put $\tilde{\xi}(a) \triangleq x(a)$ and $\tilde{\xi}(b) = x(b)$. Then $\tilde{\xi}$ is another g–convex function $\leq x$. Since $\breve{x}$ is the largest of these functions, this yields in particular $x(a) = \xi(a) \leq \breve{x}(a) \leq x(a)$, i.e., $x(a) = \breve{x}(a)$. Analogously, one finds $x(b) = \breve{x}(b)$. The property $\breve{x} \leq x_*$ on (a, b) holds, since $\breve{x}$ is dominated by x and continuous on this set.

(ii) Consider $t \in [a, b]$ with $\breve{x}(t) < x_*(t)$. We shall show that $t \notin \operatorname{supp} d\breve{l}$. To this end, we note first that, by assumption on t, there are real numbers $c, \delta > 0$ such that

$$\breve{x}(s) + c \leq x(s) \quad \text{for all} \quad s \in [t - \delta, t + \delta]\,.$$

For $0 < h \leq \delta$, consider the function x^h defined by $x^h \triangleq \breve{x}$ on $[a, t - h] \cup [t + h, b]$ and

$$x^h(s) \triangleq \breve{x}(t - h) + \int_{t-h}^{s} g(v, l^h)\, dv \quad \text{for} \quad s \in (t - h, t + h)$$

where $l^h \in \mathbb{R}$ is the unique constant satisfying

$$\breve{x}(t - h) + \int_{t-h}^{t+h} g(v, l^h)\, dv = \breve{x}(t + h)\,.$$

As $\breve{x}$ is g–convex, we have $\breve{x} \leq x^h$ on $[t - h, t + h]$ and, hence, on all of $[a, b]$.

Since $\sup_{[t-h,t+h]} x^h$ depends continuously on h through $\breve{x}(t \pm h)$ and because $\breve{x} + c \leq x$ on $[t - \delta, t + \delta]$, we may choose $h > 0$ small enough to ensure $x^h \leq x$ on this interval and, hence, even on $[a, b]$. By construction, such an x^h is a g–convex function dominated by x, and therefore also dominated by $\breve{x}$.

Altogether, we find that x^h must in fact coincide with $\breve{x}$. This implies $\breve{l} \equiv l^h$ on $[t - h, t + h]$ and, in particular, $t \notin \operatorname{supp} d\breve{l}$.

(iii) We know already that $\breve{x}$ is absolutely continuous on the open interval (a, b). Thus, in order to establish this property on all of $[a, b]$, it suffices to show that $\breve{x}$ is continuous in a and in b. The argument for a being similar, we restrict ourselves to show continuity of $\breve{x}$ in b. By Proposition 3.8, $\lim_{t \to b} \breve{x}(t)$ exists and is $\leq \breve{x}(b) = x(b)$.

If $b \in \operatorname{supp} d\breve{l}$ with $\breve{l}$ as in (ii), there is a sequence of points $t_n \in \operatorname{supp} d\breve{l} \subset (a, b)$ which increases to b. By (ii), we thus have

$$\lim_{t \to b} \breve{x}(t) = \lim_{n} \breve{x}(t_n) = \lim_{n} x_*(t_n) \geq \liminf_{t \to b} x_*(t) = x_*(b) = x(b)$$

which establishes the converse inequality.

If $b \notin \operatorname{supp} d\breve{l}$, then $b > \tau \stackrel{\Delta}{=} \sup \operatorname{supp} d\breve{l}$ and

$$\breve{x}(t) = \breve{x}(s) + \int_s^t g(v, \breve{l}(\tau))\, dv \quad \text{for all} \quad \tau \le s \le t < b\,.$$

Thus, in case $\lim_{t \to b} \breve{x}(t) < x(b) = x_*(b)$, there is a constant $c > 0$ such that for $s < b$ large enough we have $\breve{x}(t) + c \le x(t)$ for all $t \in [s, b)$. This, however, contradicts the maximality of $\breve{x}$ over $[s, b)$.

(iv) Consider $s \in (a, b)$ and $t_1, t_2 \in (a, s]$ with $t_1 < t_2$. Let

$$u \stackrel{\Delta}{=} \inf\{t \ge s \mid \breve{x}^{t_1}(t) = \breve{x}^{t_2}(t)\} \in [s, b]\,.$$

Clearly, $\breve{x}^{t_1} \le \breve{x}^{t_2}$. Moreover, either by continuity of convex envelopes (if $u < b$) or by (i) (in case $u = b$), we have $\breve{x}^{t_1}(u) = \breve{x}^{t_2}(u)$ which gives us $\breve{x}^{t_1} = \breve{x}^{t_2}$ on $[u, b]$.

We see on the one hand that $\breve{x}^{t_1}(s) \ge \breve{x}^{t_2}(s)$ is equivalent to $u = s$, and that, therefore, $\partial^+ \breve{x}^{t_1}(s) = \partial^+ \breve{x}^{t_2}(s)$ in this case.

If, on the other hand, $\breve{x}^{t_1}(s) < \breve{x}^{t_2}(s)$ or, equivalently, if $u > s$, then let $G(s, .) \stackrel{\Delta}{=} (g(s, .))^{-1}$ and define, for $x_1, x_2 \in \mathbb{R}$, the number $\rho(x_1, x_2)$ as the unique constant satisfying

$$x_1 + \int_s^u g(v, \rho)\, dv = x_2\,.$$

Using, respectively, part (ii) of this proposition, the strict monotonicity of $\rho(., x_2)$ together with $\breve{x}^{t_1}(u) = \breve{x}^{t_2}(u)$, and finally the g–convexity of $\breve{x}^{t_2}$, we obtain the series of (in-)equalities

$$G(s, \partial^+ \breve{x}^{t_1}(s)) = \rho(\breve{x}^{t_1}(s), \breve{x}^{t_1}(u)) > \rho(\breve{x}^{t_2}(s), \breve{x}^{t_2}(u)) \ge G(s, \partial^+ \breve{x}^{t_2}(s))$$

which yields $\partial^+ \breve{x}^{t_1}(s) > \partial^+ \breve{x}^{t_2}(s)$ as was to be shown.

□

3.3.2 Solution under Certainty

After these technical preliminaries, we are now in a position to describe the solution to our deterministic representation problem (3.7) in terms of inhomogeneously convex envelopes.

Theorem 3.2 *Under Assumption 3.1, any lower–semicontinuous function $x : [0, \hat{T}] \to \mathbb{R}$ with $x(\hat{T}) = 0$ permits a representation*

$$x(s) = \int_s^{\hat{T}} f(t, \sup_{s \le v \le t} l(v))\, dt \quad (0 \le s \le \hat{T}) \tag{3.12}$$

where $l : [0, \hat{T}) \to \mathbb{R} \cup \{-\infty\}$ *is a uniquely determined upper–semicontinuous function such that the above integrand* $f(., \sup_{s \le v \le .} l(v))$ *is Lebesgue–integrable over* $[s, \hat{T}]$ *for every* $s \in [0, \hat{T}]$.

This function l *is given by*

$$-f(s, l(s)) = \left(\partial^+ \breve{x}^s\right)(s) \quad (0 \le s < \hat{T}) \tag{3.13}$$

where $\breve{x}^s$ *denotes the* $(-f)$*–convex envelope of the restriction* $x|_{[s,\hat{T}]}$.

Remarkably, the converse of Theorem 3.2 is also true:

Theorem 3.3 *Under Assumption 3.1, any function* x *which may be represented as in Theorem 3.2 is lower–semicontinuous.*

Let us now prove the preceding results. We start with the

Proof of Theorem 3.2

1. The uniqueness of a function l with (3.12) has been established even more generally in Section 3.2.

2. Let us next show that l with (3.13) indeed satisfies (3.12) for any $s \in [0, \hat{T}]$. For ease of notation, we put $g \stackrel{\Delta}{=} -f$ and we define $G(t, .)$ as the inverse $G(t, .) \stackrel{\Delta}{=} (g(t, .))^{-1}$.

   Note first that lower–semicontinuity of x implies lower–semicontinuity of any restriction $x^t \stackrel{\Delta}{=} x|_{[t,\hat{T}]}$ $(t \in [0, \hat{T}])$. Any x^t is bounded from below and, therefore, has a real–valued g–convex envelope $\breve{x}^t$ by Proposition 3.9. Moreover, we may apply Proposition 3.11 (i) and (iii) to write

   $$x(s) = \breve{x}^s(s) - \breve{x}^s(\hat{T}) = -\int_s^{\hat{T}} \left(\partial^+ \breve{x}^s\right)(t)\, dt = \int_s^{\hat{T}} f\left(t, G\left(t, \left(\partial^+ \breve{x}^s\right)(t)\right)\right)\, dt\,.$$

   Thus, it suffices to show that, for all $t \in (s, \hat{T})$, we have

   $$G\left(t, \left(\partial^+ \breve{x}^s\right)(t)\right) = \sup_{s \le v \le t} G\left(v, \left(\partial^+ \breve{x}^v\right)(v)\right)\,. \tag{3.14}$$

   By Proposition 3.11 (iv), $\partial^+ \breve{x}^s(v)$ is decreasing in $s \in [0, v]$. Thus,

   $$G(v, \partial^+ \breve{x}^v(v)) \le G(v, \partial^+ \breve{x}^s(v))$$

   which, in turn, is $\le G(t, \partial^+ \breve{x}^s(t))$ for all $v \le t$ by g–convexity of $\breve{x}^s$. This proves

   $$\sup_{s \le v \le t} G\left(v, \left(\partial^+ \breve{x}^v\right)(v)\right) \le G\left(t, \left(\partial^+ \breve{x}^s\right)(t)\right)\,. \tag{3.15}$$

For the converse inequality consider the set

$$\mathcal{V} \triangleq \{v \in [s,t] \mid \breve{x}^s(v) = x(v)\}$$

and let $v^* \triangleq \sup \mathcal{V}$. We claim that

$$\breve{x}^s|_{[v^*,\hat{T}]} = \breve{x}^{v^*} . \tag{3.16}$$

For this it suffices to show that $\breve{x}^s(v^*) = x(v^*)$. To this end, let v_n $(n = 1, 2, \ldots)$ be a sequence in $\mathcal{V}$ which converges to v^*. Using the continuity of $\breve{x}^s$ and the lower–semicontinuity of x, we obtain

$$\breve{x}^s(v^*) = \lim_n \breve{x}^s(v_n) = \lim_n x(v_n) \geq \liminf_{v \to v^*} x(v) = x(v^*) \geq \breve{x}^s(v^*) .$$

Consequently, equality must hold everywhere in this line and this proves our claim (3.16).

Now, applying first Proposition 3.11 (ii) and then our cliam (3.16), we see that

$$\begin{aligned} G\left(t, \left(\partial^+ \breve{x}^s\right)(t)\right) &= G\left(v^*, \left(\partial^+ \breve{x}^t\right)(v^*)\right) \\ &= G\left(v^*, \left(\partial^+ \breve{x}^{v^*}\right)(v^*)\right) \leq \sup_{s \leq v \leq t} G\left(v, \left(\partial^+ \breve{x}^v\right)(v)\right) . \end{aligned} \tag{3.17}$$

Together, inequalities (3.15) and (3.17) imply (3.14).

3. It remains to establish the upper–semicontinuity of l. From Proposition 3.11 (iv) we infer that

$$\partial^+ \breve{x}^s(t) \geq \partial^+ \breve{x}^t(t)$$

for any $t > s$. Letting $t \downarrow s$, the left side of this inequality converges to $\partial^+ \breve{x}^s(s)$, while its right side is in the limit not larger than $\limsup_{t \downarrow s} \partial^+ \breve{x}^t(t)$. Thus, we have

$$l(s) = G(s, \partial^+ \breve{x}^s(s)) \geq G(s, \limsup_{t \downarrow s} \partial^+ \breve{x}^t(t)) = \limsup_{t \downarrow s} l(t)$$

because of the continuity and monotonicity of G. Hence, $l(.)$ is upper–semi-continuous from the right.

Now, consider $t < s$ and let $u \in (s, \hat{T})$. Since $\breve{x}^t$ is g–convex with $\breve{x}^t(t) = x(t)$, we have

$$l(t) = G(t, \partial^+ \breve{x}^t(t)) \leq \rho(t, u, \breve{x}^t(u) - x(t)) ,$$

where $\rho(t, u, \Delta) \in \mathbb{R}$ is the unique constant with

$$\int_t^u g(v, \rho)\, dv = \Delta .$$

As $\rho(t,s,\Delta)$ is continuous in (t,s,Δ) and increasing in Δ, this inequality yields

$$\limsup_{t\uparrow s} l(t) \le \rho(s,u,\limsup_{t\uparrow s}\{\breve{x}^t(u)-x(t)\})\,.$$

Using $\breve{x}^t \le \breve{x}^s$ on $[s,\hat{T}]$ and the lower–semicontinuity of x, we derive the estimate

$$\limsup_{t\uparrow s} l(t) \le \rho(s,u,\breve{x}^s(u)-x(s)) = \rho(s,u,\breve{x}^s(u)-\breve{x}^s(s))\,.$$

Due to the g–convexity of $\breve{x}^s$, the last expression decreases to $G(s,\partial^+\breve{x}^s(s)) = l(s)$ as $u \downarrow s$. This yields that we also have upper–semicontinuity from the left.

□

Let us now turn to the

Proof of Theorem 3.3 Define

$$i_s(t) \stackrel{\Delta}{=} 1_{(s,\hat{T}]}(t)\, f(t, \sup_{s\le v\le t} l(v))$$

such that $x(s) = \int_0^{\hat{T}} i_s(t)\,dt$ for all $s \in [0,\hat{T}]$. Obviously,

$$i_s(t) \ge 0 \wedge f(t, \sup_{0\le v\le t} l(v)) \in L^1([0,\hat{T}],\,dt) \tag{3.18}$$

for every $s \in [0,\hat{T}]$, i.e., the family of integrands $(i_s(.),\ s \in [0,\hat{T}])$ is bounded from below by some Lebesgue–integrable function.

Now, let us show that $x(s) = \int_0^{\hat{T}} i_s(t)\,dt$ is lower–semicontinuous at each point $s^* \in [0,\hat{T}]$. Indeed, on the one hand, we have

$$\lim_{s\downarrow s^*} i_s(t) = 1_{(s^*,\hat{T}]}(t)\, f(t, \sup_{s^*<v\le t} l(v)) \quad \text{for every} \quad t \in [0,\hat{T})\,,$$

and, because of estimate (3.18), we may use Fatou's lemma to obtain

$$\liminf_{s\downarrow s^*} x(s) \ge \int_0^{\hat{T}} \lim_{s\downarrow s^*} i_s(t)\,dt = \int_{s^*}^{\hat{T}} f(t, \sup_{s^*<v\le t} l(v)) \ge x(s^*)\,.$$

On the other hand, we have

$$\lim_{s\uparrow s^*} i_s(t) = 1_{[s^*,\hat{T}]}(t)\, f(t, \sup_{s^*\le v\le t} l(v)) \quad \text{for all} \quad t \in [0,\hat{T}]$$

since $l(.)$ is upper–semicontinuous. Thus, by Fatou's lemma again,

$$\liminf_{s\uparrow s^*} x(s) \ge \int_0^{\hat{T}} \lim_{s\uparrow s^*} i_s(t)\,dt = \int_{s^*}^{\hat{T}} f(t, \sup_{s^*\le v\le t} l(v))\,dt = x(s^*)\,.$$

Hence, $\liminf_{s\to s^*} x(s) \ge x(s^*)$ as we wanted to show. □

3.3.3 Explicit Solution in the Locally Convex Case

Theorem 3.2 reduces our deterministic representation problem (3.7) to the problem of computing the initial righthand derivatives of all g–convex envelopes $\breve{x}^t$ $(t \in [0, \hat{T}))$ where $g \overset{\Delta}{=} -f$. This task is trivial if x itself is g–convex. The following lemma shows, how we can treat the case when x is locally g–convex.

Lemma 3.12 *Consider a function $x : [0, \hat{T}] \to \mathbb{R}$ for which each restriction $x|_{[0,T]}$ $(T \in [0, \hat{T}))$ is g–convex on its domain $[0, T]$.*

(i) There is a unique point in time $t^* \in [0, \hat{T}]$ such that, for any $t \in [0, \hat{T}]$, the g–convex envelope $\breve{x}^t$ of the restriction $x|_{[t,\hat{T}]}$ satisfies

$$\{\breve{x}^t = x\} = [t, t^* \vee t] \cup \{\hat{T}\}\,. \tag{3.19}$$

We have $t^* < \hat{T}$ iff $x(\hat{T}-) > x(\hat{T})$.

(ii) Relation (3.19) entails

$$\partial^+ \breve{x}^t(t) = \begin{cases} \partial^+ x(t) & \text{for } t \in [0, t^*)\,, \\ g(t, l_{t,\hat{T}}) & \text{for } t \in [t^*, \hat{T})\,. \end{cases} \tag{3.20}$$

where $l_{t,\hat{T}} \in \mathbb{R}$ is the unique constant satisfying

$$x(\hat{T}) - x(t) = \int_t^{\hat{T}} g(s, l_{t,\hat{T}})\, ds \tag{3.21}$$

(iii) We have

$$\partial^+ x(t) \begin{array}{c} \leq \\ \geq \end{array} g(t, l_{t,\hat{T}}) \quad \text{for} \quad \begin{array}{c} t \in [0, t^*) \\ t \in (t^*, \hat{T}] \end{array}$$

and these inequalities are strict if the restrictions $x|_{[0,T]}$ $(T \in [0, \hat{T}))$ are strictly g–convex.

For $t = t^*$, we have $\partial^+ x(t^*) \geq g(t^*, l_{t^*,\hat{T}})$ with equality in case $\Delta\partial^+ x(t^*) = 0$. In particular, the mapping $t \mapsto \partial^+ \breve{x}^t(t)$ is continuous if x is continuously differentiable on $[0, \hat{T})$.

PROOF :

1. To prove existence of t^* with (3.19), we first consider the special case $t = 0$. In the first place, there can be at most one point t^* satisfying our assertion since necessarily
$$\{\breve{x}^0 = x\} = [0, t^*] \cup \{\hat{T}\}\,,$$

i.e.,

$$t^* = \sup\{s \in [0, \hat{T}) \mid \breve{x}^0(s) = x(s)\}\,.$$

We thus define t^* by this relation and show next that it has the desired property (3.19) for $t = 0$.

To this end, note that $x(t^*) = \breve{x}^0(t^*)$. Indeed, in case $t^* < \hat{T}$, this follows by continuity of x and $\breve{x}^0$ on $[0, \hat{T})$. In case $t^* = \hat{T}$ this identity is due to Proposition 3.11 (i). It yields that $\breve{x}^0|_{[0,t^*]}$ is the g–convex envelope of the restriction $x|_{[0,t^*]}$. As x is locally g–convex by assumption, this means that indeed $\breve{x}^0 = x$ on $[0, t^*]$ which is our claim (3.19) for $t = 0$.

By definition, $t^* = \hat{T}$ iff $\breve{x}^0 = x$, i.e., iff x is g–convex on the whole interval $[0, \hat{T}]$. Due to the local g–convexity of x, this is equivalent to $x(\hat{T}) \geq x(\hat{T}-)$ by Proposition 3.8. Observe that existence of the limit $x(\hat{T}-)$ follows by the same arguments as for the '$(ii) \Rightarrow (iii)$'–part of the latter proposition.

2. For $t < t^*$, we have $\breve{x}^t = \breve{x}^0|_{[t,\hat{T}]}$ by Step 1 and, therefore,

$$\{\breve{x}^t = x\} = \{\breve{x}^0 = x\} \cap [t, \hat{T}] = [t, t^*] \cup \{\hat{T}\}\,.$$

For $t \geq t^*$, we show that $\{s \in (t, \hat{T}) \mid \breve{x}^t(s) = x(s)\} = \varnothing$. Suppose to the contrary that there is some $s \in (t, \hat{T})$ with $\breve{x}^t(s) = x(s)$. By local g–convexity of x, this yields $\breve{x}^t = x$ on $[t, s]$ and, therefore,

$$\tilde{x} \triangleq \begin{cases} x & \text{on } [0, s] \\ \breve{x}^t & \text{on } [t, \hat{T}] \end{cases}$$

is a well–defined function $[0, \hat{T}] \to \mathbb{R}$. In fact, since $t < s$, it is a smooth pasting of two g–convex functions and, therefore, it is g–convex on its domain, too. As obviously $\tilde{x} \leq x$, this yields, on the one hand, $\tilde{x} \leq \breve{x}^0$ by definition of the convex envelope. On the other hand, $\breve{x}^t \geq \breve{x}^0|_{[t,\hat{T}]}$ yields $\tilde{x} \geq \breve{x}^0$. Hence, we find $\tilde{x} = \breve{x}^0$ and, in particular, $[0, s] \subset \{\breve{x}^0 = x\}$ — a contradiction to $s > t^* = \sup(\{\breve{x}^0 = x\} \cap [0, \hat{T}))$. This completes the proof of assertion (i).

3. Assertion (i) implies that, for $t \in [0, t^*)$, $\breve{x}^t$ and x coincide on the non–trivial interval $[t, t^*]$. Hence,

$$\partial^+ \breve{x}^t(t) = \partial^+ x(t) \quad \text{for all} \quad t \in [0, t^*)\,,$$

which is the first part of assertion (ii).

For $t \in [t^*, \hat{T})$, assertion (i) yields that $x|_{[t,\hat{T}]}$ and its envelope $\breve{x}^t$ only coincide at time t and at time $\hat{T}$. It thus follows from Proposition 3.11 (ii) that

$$\partial^+ \breve{x}^t(t) = g(t, l_{t,\hat{T}})$$

for these times, and this establishes the second part of assertion (ii).

4. Let us now turn to assertion (iii). As x is locally g–convex, there is a rightcontinuous, increasing function $l^x : (0,\hat{T}) \to \mathbb{R}$ such that

$$\partial^+ x(t) = g(t, l^x(t)) \quad \text{for all} \quad t \in (0,\hat{T})\,.$$

We have to show that $l^x(t) \le l_{t,\hat{T}}$ for $t \in [0,t^*)$ and $l^x(t) \ge l_{t,\hat{T}}$ for $t \in (t^*,\hat{T})$ with strict inequality if x is locally g–convex in the strict sense.

To this end, let us denote by $l^0_{s,t}$ the unique constant such that

$$\breve{x}^0(t) - \breve{x}^0(s) = \int_s^t g(v, l^0_{s,t})\,dv \quad (0 \le s < t \le \hat{T})\,.$$

For the first inequality in (iii), note that (strict) g–convexity of $x|_{[0,t^*]}$ implies $l^x(t) \le (<)\ l^x_{t,t^*}$. Since $x = \breve{x}^0$ on $[0,t^*]$, we have $l^x_{t,t^*} = l^0_{t,t^*}$. By (global) g–convexity of $\breve{x}^0$, this in turn is $\le l^0_{t,\hat{T}} = l^x_{t,\hat{T}}$ where the preceding equality follows from $x(\hat{T}) = \breve{x}^0(\hat{T})$ and $x(t) = \breve{x}^0(t)$ for $t < t^*$. Thus, we have shown

$$l^x(t) \le (<) l^x_{t,t^*} = l^0_{t,t^*} \le l^0_{t,\hat{T}} = l^x_{t,\hat{T}} \quad \text{for} \quad t \in [0,t^*)$$

which establishes the first inequality in (iii).

For the second inequality, consider $t > t^*$ and use (strict) local g–convexity of x to estimate $l^x(t) \ge (>)\ l^x(t^*) = \inf_{s\in(t^*,\hat{T})} l_{t^*,s}$. Since $x \ge \breve{x}^0$ with equality in t^*, the preceding infimum is $\ge \inf_{s\in(t^*,\hat{T})} l^0_{t^*,s}$. From Proposition 3.11 (ii) and the definition of t^*, we infer that $l^0_{t^*,s} \equiv l^0_{t,\hat{T}} \equiv l^0_{t^*,\hat{T}}$ for all times $s,t \in [t^*,\hat{T}]$. In addition, $x \ge \breve{x}^0$ and $x(\hat{T}) = \breve{x}^0(\hat{T})$ entail $l^0_{t,\hat{T}} \ge l_{t,\hat{T}}$. Altogether, the preceding considerations yield

$$l^x(t) \ge (>) \inf_{s\in(t^*,\hat{T})} l_{t^*,s} \ge l^0_{t^*,\hat{T}} = l^0_{t,\hat{T}} \ge l_{t,\hat{T}}$$

which proves the second inequality in (iii).

As x is continuous on $[0,\hat{T})$, also $t \mapsto l_{t,\hat{T}}$ is continuous on this interval. Since l^x is rightcontinuous, the second inequality yields $l^x(t^*) \ge l_{t^*,\hat{T}}$ in general. By monotonicity of l^x we obtain from the first inequality that $l_{t^*,\hat{T}} \ge l^x(t^*-)$. Hence, we have $l_{t^*,\hat{T}} = l^x(t^*)$ and, therefore, $g(t^*, l_{t,\hat{T}}) = \partial^+ x(t^*)$ provided $l^x(t^*-) = l^x(t^*)$, i.e., if $\Delta\partial^+ x(t^*) = 0$.

□

3.4 Existence in the General Stochastic Case

Let us now focus on the general case and prove existence of a solution to our stochastic representation problem (3.1). Our main result is

Theorem 3.4 *Suppose that f satisfies Assumption 3.1 and that, furthermore, the mapping $f(t, .) : \mathbb{R} \to \mathbb{R}$ is continuously differentiable for every $t \in [0, \hat{T}]$. Let X be a non-negative optional process, bounded from above by some martingale. Assume in addition that X is lower–semicontinuous in expectation with $X(\hat{T}) = 0$.*

Then there exists an optional solution L to our representation problem (3.1)*, i.e., there is an optional process L such that*

$$X(S) = \mathbb{E}\left[\int_S^{\hat{T}} f(t, \sup_{S \le v \le t} L(v))\, dt \,\middle|\, \mathcal{F}_S\right]$$

for every stopping time $S \in \mathcal{S}$.

The proof of this theorem will be given in Section 3.4.2 below. The following Section 3.4.1 provides some technical preliminaries. Corollary 4.11 in the next chapter provides an application of this result to the utility maximization problem studied in the preceding chapter.

Remark 3.13 *An optional process X of class (D) is called lower–semicontinuous in expectation if, for every $S \in \mathcal{S}$, we have*

$$\liminf_n \mathbb{E}X(S_n) \ge \mathbb{E}X(S)$$

whenever $(S_n,\ n = 1, 2, \ldots)$ is a monotone sequence of stopping times converging to S almost surely.

3.4.1 Auxiliary Optimal Stopping Problems of Gittins–Type

In this section, we will provide some preliminaries for the proof of Theorem 3.4. As a motivation, let us briefly describe the original background of these technicalities.

Gittins' Problem of Optimal Dynamic Scheduling

The Gittins–problem amounts to finding an optimal schedule for a certain number of independent projects. When worked on, each of these projects accrues a specific stochastic reward. The aim is to subsequently work on the given projects so as to maximize the total expected reward.

Gittins' celebrated idea to solve this high–dimensional optimization problem was to introduce a family of simpler benchmark problems which allowed him to define a performance measure — later called Gittins–index — for each of the original projects. An optimal schedule could then be given in form of an index–rule: "always work on a project with maximal Gittins–index".

To describe the connection between the Gittins–index and our representation problem (3.1), let us review some of the results on the auxiliary benchmark problems which can be found in El Karoui and Karatzas (1994). These authors consider a project whose reward at time t is given by some stochastic rate $h(t) > 0$. With this project, they associate the family of optimal stopping problems

$$V(s,m) \stackrel{\Delta}{=} \operatorname*{ess\,sup}_{T \in \mathcal{S}(s)} \mathbb{E}\left[\int_s^T e^{-\alpha(t-s)} h(t)\, dt + m e^{-\alpha(T-s)} \,\middle|\, \mathcal{F}_s \right] \quad (s, m \geq 0)\,. \tag{3.22}$$

The constant m is interpreted as a reward–upon–stopping, the optimization starts at time s, and $\alpha > 0$ is a constant discount rate.

El Karoui and Karatzas (1994) show that, under appropriate conditions, the Gittins–index $M(s)$ of this project at time s can, loosely speaking, be described as the minimal reward–upon–stopping such that immediate termination of the project is optimal in the auxiliary stopping problem (3.22):

$$M(s) = \inf\{m \geq 0 \mid V(s,m) = m\} \quad (s \geq 0)\,. \tag{3.23}$$

Without making further use of it, they also establish the alternative representation

$$M(s) = \operatorname*{ess\,sup}_{T \in \mathcal{S}^{>}(s)} \frac{\mathbb{E}\left[\int_s^T e^{-\alpha t} h(t)\, dt \,\middle|\, \mathcal{F}_s \right]}{\mathbb{E}\left[\int_s^T \alpha e^{-\alpha t}\, dt \,\middle|\, \mathcal{F}_s \right]} \quad (s \geq 0) \tag{3.24}$$

which is provided as equation (3.11) in their Proposition 3.4. Note that this identity becomes precisely our equation (3.5) which characterizes the solution L to the representation problem (3.1) in the special case where $\hat{T} \stackrel{\Delta}{=} +\infty$ and where

$$f(t,l) \stackrel{\Delta}{=} -\alpha e^{-\alpha t} l, \qquad X(t) \stackrel{\Delta}{=} -\mathbb{E}\left[\int_t^{+\infty} e^{-\alpha s} h(s)\, ds \,\middle|\, \mathcal{F}_t \right] \quad (t \geq 0,\ l \in \mathbb{R})\,.$$

Moreover, in their equation (3.7) El Karoui and Karatzas (1994) note the identity

$$\mathbb{E}\left[\int_s^{+\infty} e^{-\alpha t} h(t)\, dt \,\middle|\, \mathcal{F}_s \right] = \mathbb{E}\left[\int_s^{+\infty} \alpha e^{-\alpha t} \sup_{s \leq v \leq t} M(v)\, dt \,\middle|\, \mathcal{F}_s \right] \quad (s \geq 0)\,. \tag{3.25}$$

For the above choices of $\hat{T}$, f, and X, this transforms into our backward formulation

$$X(s) = \mathbb{E}\left[\int_s^{\hat{T}} f(t, \sup_{s \leq v \leq t} M(v))\, dt \,\middle|\, \mathcal{F}_s \right] \quad (s \geq 0)$$

of the representation problem. Thus, in this special case, the Gittins–index M for the project with rewards $(h(t),\ t \geq 0)$ coincides with the solution L to our representation problem (3.1). Observe, however, that El Karoui and Karatzas consider identity (3.25) merely as a property of the Gittins–index M and not as a characterization of M as the solution to a representation problem.

The key point for our subsequent analysis is that the results by El Karoui and Karatzas indicate an alternative possibility to construct such a representation, namely by using the formula (3.23). In contrast to formula (3.4) in our Uniqueness Theorem 3.1, this representation rests upon the value function of a *standard* optimal stopping problem. This allows us to apply the well established 'théorie generale' of Snell–envelopes. In fact, as we shall see, an approach based on Snell–envelopes allows us to establish the desired representation (3.1) for a large class of optional processes X and functions f under a finite time horizon $\hat{T} < +\infty$.

In order to describe the Snell–envelopes which are most suitable for our purposes, we pass from the maximization problem (3.22) to a minimization problem. More precisely, the reward–upon–stopping $me^{-\alpha(T-s)}$ is transformed into the accumulated costs–of–continuation $\int_s^T f(t,l)\,dt$, and the accumulated discounted reward of working on the project $\int_s^T e^{-\alpha t}h(t)\,dt$ is replaced by the costs $X(T)$. Hence, problem (3.22) is transferred into the family of optimal stopping problems

$$Y^l(S) = \mathbb{P}\text{–}\operatorname*{ess\,inf}_{T \in \mathcal{S}(S)} \mathbb{E}\left[X(T) + \int_S^T f(t,l)\,dt \,\middle|\, \mathcal{F}_S \right] \quad (S \in \mathcal{S},\ l \in \mathbb{R})\,. \tag{3.26}$$

The Crucial Lemma

The following Lemma 3.14 analyzes the family of auxiliary Gittins–problems (3.26). Its most important assertion (iii) provides a version of the Envelope Theorem which describes the marginal increase of minimal costs $\partial_l Y^l(S)$ when 'instantaneous costs of continuation' increase from $f(t,l)$ to $f(t,l+dl)$. This description will be a crucial part for our proof of Theorem 3.4 in the following section. The key idea for this strategy of proof is essentially due to Nicole El Karoui who pointed out the usefulness of the Envelope Theorem as a means to relate her results on Gittins–problems to our representation problem.

Lemma 3.14 *Under the assumptions of Theorem 3.4, there is a product–measurable mapping*

$$\begin{aligned} Y : \Omega \times [0,\hat{T}] \times \mathbb{R} &\to \mathbb{R} \\ (\omega,t,l) &\mapsto Y^l(\omega,t) \end{aligned}$$

with the following properties:

(i) For $l \in \mathbb{R}$ fixed, $Y^l : \Omega \times [0, \hat{T}] \to \mathbb{R}$ is an optional process such that

$$Y^l(S) = \operatorname*{ess\,inf}_{T \in \mathcal{S}(S)} \mathbb{E}\left[X(T) + \int_S^T f(t,l)\,dt \,\middle|\, \mathcal{F}_S \right] \quad \mathbb{P}\text{–}a.s. \tag{3.27}$$

for every stopping time $S \in \mathcal{S}$. Moreover, the stopping time

$$T_S^l \triangleq \inf\left\{ t \geq S \;\middle|\; Y^l(t) = X(t) \right\} \leq \hat{T}$$

is optimal in (3.27), i.e.,

$$Y^l(S) = \mathbb{E}\left[X(T_S^l) + \int_S^{T_S^l} f(t,l)\,dt \,\middle|\, \mathcal{F}_S \right].$$

(ii) For fixed $(\omega, s) \in \Omega \times [0, \hat{T}]$, the mapping $l \mapsto Y^l(\omega, s)$ is continuously decreasing from

$$Y^{-\infty}(\omega, s) \triangleq \lim_{l \downarrow -\infty} Y^l(\omega, s) = X(\omega, s).$$

In particular, there is a continuous extension of Y to $\Omega \times [0, \hat{T}] \times (\mathbb{R} \cup \{-\infty\})$.

(iii) For every stopping time $S \in \mathcal{S}$, the mapping $l \mapsto Y^l(\omega, S(\omega))$ is absolutely continuous for $\mathbb{P}$–a.e. $\omega \in \Omega$. A version of its density is given by

$$\partial_l Y^l(\omega, S(\omega)) = \mathbb{E}\left[\int_S^{T_S^l} \partial_l f(t,l)\,dt \,\middle|\, \mathcal{F}_S \right](\omega) \quad (l \in \mathbb{R}) \tag{3.28}$$

where the above conditional expectations are chosen in a product–measurable way.

The proof of the preceding lemma is rather lengthy and technical. We therefore split it up in several parts and start with some

Preliminaries Due to our assumptions on X, we may apply Théorème 2.28 in El Karoui (1981) to obtain existence of optional processes $\tilde{Y}^l$ ($l \in \mathbb{R}$) such that

$$\tilde{Y}^l(S) = \operatorname*{ess\,inf}_{T \in \mathcal{S}(s)} \mathbb{E}\left[X(T) + \int_S^T f(t,l)\,dt \,\middle|\, \mathcal{F}_S \right] \leq X(S)$$

for every stopping time $S \in \mathcal{S}$ and every $l \in \mathbb{R}$. Moreover, Théorème 2.41 in El Karoui (1981) implies that, for $S \in \mathcal{S}$ fixed,

$$\tilde{T}_S^l \triangleq \inf\left\{ t \geq S \;\middle|\; \tilde{Y}^l(t) = X(t) \right\} \leq \hat{T}$$

is optimal in the sense that

$$\tilde{Y}^l(S) = \mathbb{E}\left[X(\tilde{T}^l_S) + \int_S^{\tilde{T}^l_S} f(t,l)\,dt \,\middle|\, \mathcal{F}_S \right].$$

For l, $l' \in \mathbb{R}$ with $l \le l'$, the monotonicity of $f(t,.)$ $(0 \le t \le \hat{T})$ yields

$$\mathbb{E}\left[X(T) + \int_S^T f(t,l')\,dt \,\middle|\, \mathcal{F}_S \right] \le \mathbb{E}\left[X(T) + \int_S^T f(t,l)\,dt \,\middle|\, \mathcal{F}_S \right]$$

for all $T \in \mathcal{S}(S)$. As $\tilde{Y}^{l'}(S)$ (resp. $\tilde{Y}^l(S)$) is the essential infimum of the left (resp. the right) side of this inequality where T ranges over $\mathcal{S}(S)$, this implies

$$\tilde{Y}^{l'}(S) \le \tilde{Y}^l(S) \quad \mathbb{P}\text{–a.s.} \tag{3.29}$$

In addition, we have

$$\begin{aligned}
\tilde{Y}^l(S) &\le \mathbb{E}\left[X(\tilde{T}^{l'}_S) + \int_S^{\tilde{T}^{l'}_S} f(t,l)\,dt \,\middle|\, \mathcal{F}_S \right] \\
&= \mathbb{E}\left[X(\tilde{T}^{l'}_S) + \int_S^{\tilde{T}^{l'}_S} f(t,l')\,dt \,\middle|\, \mathcal{F}_S \right] + \mathbb{E}\left[\int_S^{\tilde{T}^{l'}_S} \{f(t,l) - f(t,l')\}\,dt \,\middle|\, \mathcal{F}_S \right] \\
&= \tilde{Y}^{l'}(S) + \mathbb{E}\left[\int_S^{\tilde{T}^{l'}_S} \{f(t,l) - f(t,l')\}\,dt \,\middle|\, \mathcal{F}_S \right]
\end{aligned}$$

where the last equality follows from optimality of $\tilde{T}^{l'}_S$. For $l \le l'$, we have $f(t,l) - f(t,l') \ge 0$ on $[0,\hat{T}]$, and thus the preceding estimate yields

$$\tilde{Y}^l(S) \le \tilde{Y}^{l'}(S) + \int_0^{\hat{T}} |f(t,l') - f(t,l)|\,dt \quad \mathbb{P}\text{–a.s.} \tag{3.30}$$

Since both estimates (3.29) and (3.30) hold true for every stopping time $S \in \mathcal{S}$, optionality of both $\tilde{Y}^l$ and $\tilde{Y}^{l'}$ entails the pathwise estimate

$$\tilde{Y}^{l'}(s) \le \tilde{Y}^l(s) \le \tilde{Y}^{l'}(s) + \int_0^{\hat{T}} |f(t,l') - f(t,l)|\,dt \quad \text{for all} \quad s \in [0,\hat{T}] \quad \mathbb{P}\text{–a.s.}$$

by Meyer's optional section theorem. In fact, we may even choose $\tilde{Y}^l$ for $l \in \mathbb{Q}$ such that the above relation holds true simultaneously at each point $\omega \in \Omega$ for all rational $l \le l'$. Similarly, we may assume that $\tilde{Y}(\omega,t) \le X(\omega,t)$ for all $l \in \mathbb{Q}$ and any $(\omega,t) \in \Omega \times [0,\hat{T}]$.

With this choice of the auxiliary processes $\tilde{Y}^l$ $(l \in \mathbb{Q})$, we now come to the

Construction of Y and Proof of Lemma 3.14 (i) For each $l \in \mathbb{R}$, define the process

$$Y^l(s) \stackrel{\Delta}{=} \lim_{\mathbb{Q}\ni r\downarrow l} \tilde{Y}^r(s) = \inf_{l<r\in\mathbb{Q}} \tilde{Y}^r(s) \quad (s \in [0,\hat{T}]) .$$

We claim that Y^l *is indistinguishable from* $\tilde{Y}^l$ *for every* $l \in \mathbb{R}$. Indeed, Y^l is obviously optional. As $\tilde{Y}^r \geq \tilde{Y}^l$ for all rational $r > l$, we also have $Y^l \geq \tilde{Y}^l$. For the remaining converse inequality, fix $S \in \mathcal{S}$ and note that, for every $T \in \mathcal{S}(S)$,

$$\begin{aligned} Y^l(S) &= \lim_{\mathbb{Q}\ni r\downarrow l} \tilde{Y}^r(S) \\ &\leq \liminf_{\mathbb{Q}\ni r\downarrow l} \mathbb{E}\left[X(T) + \int_S^T f(t,r)\,dt \,\middle|\, \mathcal{F}_S \right] = \mathbb{E}\left[X(T) + \int_S^T f(t,l)\,dt \,\middle|\, \mathcal{F}_S \right] . \end{aligned}$$

Since this estimate holds true for all $T \in \mathcal{S}(S)$, we obtain may pass to the essential infimum on its right side to obtain $Y^l(S) \leq \tilde{Y}^l(S)$ almost surely. By optionality, this entails $Y^l(t) \leq \tilde{Y}^l(t)$ for all $t \in [0,\hat{T}]$ $\mathbb{P}$–a.s., which is the asserted converse inequality.

Now, Y^l and $\tilde{Y}^l$ being indistinguishable, *optimality of* T^l_S follows from optimality of $\tilde{T}^l_S$. This completes the proof of assertion (i).

Proof of Lemma 3.14 (ii) To prove the first part of assertion (ii), recall that we have chosen $\tilde{Y}^l$ ($l \in \mathbb{Q}$) such that

$$\tilde{Y}^{l'}(\omega,s) \leq \tilde{Y}^l(\omega,s) \leq \tilde{Y}^{l'}(\omega,s) + \int_0^{\hat{T}} |f(t,l') - f(t,l)|\,dt$$

for all $\omega \in \Omega$, $s \in [0,\hat{T}]$ and all rational $l \leq l'$. Taking rational limits, we infer from this that

$$Y^{l'}(\omega,s) \leq Y^l(\omega,s) \leq Y^{l'}(\omega,s) + \int_0^{\hat{T}} |f(t,l') - f(t,l)|\,dt$$

for all $\omega \in \Omega$, $s \in [0,\hat{T}]$ and all *real* $l \leq l'$. This inequality proves the claimed *continuity and monotonicity of* $l \mapsto Y^l(\omega,s)$.

We next show that, for $S \in \mathcal{S}$ fixed, we have $T^l_S(\omega) \leq T^{l'}_S(\omega)$ *simultaneously for all* $l \leq l'$ *and all* $\omega \in \Omega$. Indeed, by construction, we have $Y^{l'}(\omega,s) \leq Y^l(\omega,s) \leq X(\omega,s)$ for every $l' \leq l$, $s \in [0,\hat{T}]$ and all $\omega \in \Omega$. This yields

$$\{t \geq S(\omega) \mid Y^{l'}(\omega,s) = X(\omega,s)\} \subset \{t \geq S(\omega) \mid Y^l(\omega,s) = X(\omega,s)\}$$

whence $T^{l'}_S(\omega) \geq T^l_S(\omega)$ by definition of these stopping times.

To complete the proof of (ii), we next determine *the limit* $Y^{-\infty}$. By optimality of T^l_S, we have

$$X(S) \geq Y^l(S) = \mathbb{E}\left[X(T^l_S) + \int_S^{T^l_S} f(t,l)\,dt \,\middle|\, \mathcal{F}_S \right]$$

for any $l \in \mathbb{R}$. Letting $l \downarrow -\infty$, this entails

$$X(S) \geq Y^{-\infty}(S) \geq \liminf_{l \downarrow -\infty} \mathbb{E}\left[X(T_S^l) \,\middle|\, \mathcal{F}_S \right] + \mathbb{E}\left[\int_S^{T_S^{-\infty}} +\infty \, dt \,\middle|\, \mathcal{F}_S \right]. \tag{3.31}$$

Note that $T_S^{-\infty} \triangleq \lim_{l \downarrow -\infty} T_S^l$ exists because of the monotonicity of $l \mapsto T_S^l$. As X is bounded from below, Fatou's lemma yields the estimate

$$\liminf_{l \downarrow -\infty} \mathbb{E}\left[X(T_S^l) \,\middle|\, \mathcal{F}_S \right] \geq \mathbb{E}\left[\liminf_{l \downarrow -\infty} X(T_S^l) \,\middle|\, \mathcal{F}_S \right] \geq \mathbb{E}\left[X(T_S^{-\infty}) \,\middle|\, \mathcal{F}_S \right]$$

for the first summand on the right side of (3.31). Here, the second inequality follows from pathwise lower–semicontinuity of X which, for optional processes of class (D), is implied by lower–semicontinuity in expectation; see Dellacherie and Lenglart (1982). As X only takes finite values, the second summand must vanish, and, consequently, we almost surely have $T_S^{-\infty} = S$. Thus, the right side of (3.31) is $\geq X(S)$ $\mathbb{P}$–a.s. This proves $Y^{-\infty}(S) = X(S)$ almost surely.

It finally remains to prove our version of the Envelope Theorem.

Proof of Lemma 3.14 (iii) Fix $S \in \mathcal{S}$ and let $D_S^l(\omega)$ be a product–measurable version of

$$\mathbb{E}\left[\int_S^{T_S^l} \partial_l f(t,l) \, dt \,\middle|\, \mathcal{F}_S \right](\omega) \quad (\omega \in \Omega, l \in \mathbb{R}).$$

We have to show that almost surely

$$Y_S^{l^*} - Y_S^{l_*} = \int_{l_*}^{l^*} D_S^l \, dl \quad \text{for all} \quad l_* \leq l^*.$$

In fact, as both sides in this relation are jointly continuous in (l_*, l^*), it suffices to establish the above equality almost surely for $l_* \leq l^*$ fixed. By Fubini's theorem, this amounts to verifying that $Y_S^{l^*} - Y_S^{l_*}$ is a version of the conditional expectation

$$\mathbb{E}\left[\int_{l_*}^{l^*} \int_S^{T_S^l} \partial_l f(t,l) \, dt \, dl \,\middle|\, \mathcal{F}_S \right].$$

To this end, fix a set $A \in \mathcal{F}_S$ and consider a partition $\tau = \{l_* = l_0 < l_1 < \ldots < l_{n+1} = l^*\}$ of the interval $[l_*, l^*]$. Write

$$\mathbb{E}\left[\left(Y_S^{l^*} - Y_S^{l_*} \right) 1_A \right] = \sum_{i=0}^{n} \mathbb{E}\left[\left(Y_S^{l_{i+1}} - Y_S^{l_i} \right) 1_A \right]$$

and use optimality of $T_S^{l_{i+1}}$ and $T_S^{l_i}$, respectively, to estimate

$$\mathbb{E}\left[\left(Y_S^{l^*} - Y_S^{l_*}\right) 1_A\right] \geq \sum_{i=0}^{n} \mathbb{E}\left[\int_S^{T^{l_{i+1}}} \{f(t, l_{i+1}) - f(t, l_i)\}\, dt\, 1_A\right] \triangleq I^\tau \tag{3.32}$$

and

$$\mathbb{E}\left[\left(Y_S^{l^*} - Y_S^{l_*}\right) 1_A\right] \leq \sum_{i=0}^{n} \mathbb{E}\left[\int_S^{T^{l_i}} \{f(t, l_{i+1}) - f(t, l_i)\}\, dt\, 1_A\right] \triangleq II^\tau . \tag{3.33}$$

Using the Mean Value Theorem, we may rewrite I^τ as

$$\begin{aligned} I^\tau &= \sum_{i=0}^{n} \mathbb{E}\left[\int_S^{T_S^{l_{i+1}}} \partial_l f(t, \lambda_i^t)(l_{i+1} - l_i)\, dt\, 1_A\right] \\ &= \mathbb{E}\left[\int_{l_*}^{l^*} \left(\sum_{i=0}^{n} \int_S^{T_S^{l_{i+1}}} \partial_l f(t, \lambda_i^t)\, dt\, 1_{[l_i, l_{i+1})}(l)\right) dl\, 1_A\right] \end{aligned}$$

for some $\lambda_i^t \in (l_i, l_{i+1})$ $(i = 0, \ldots, n, t \in [0, \hat{T}])$.

For mesh $\|\tau\|$ tending to zero, continuity of $\partial_l f(t, .)$ for every $t \in [0, \hat{T}]$ and monotonicity of $l \mapsto T_S^l$ entail that the sum appearing in the preceding conditional expectation converges to $\int_S^{T_S^{l+}} \partial_l f(t, l)\, dt$ pointwise. By dominated convergence, this implies

$$\lim_{\|\tau\| \to 0} I^\tau = \mathbb{E}\left[\int_{l_*}^{l^*} \int_S^{T_S^{l+}} \partial_l f(t, l)\, dt\, dl\, 1_A\right] \triangleq I .$$

An analogous argument shows

$$\lim_{\|\tau\| \to 0} II^\tau = \mathbb{E}\left[\int_{l_*}^{l^*} \int_S^{T_S^{l-}} \partial_l f(t, l)\, dt\, dl\, 1_A\right] \triangleq II .$$

For every $\omega \in \Omega$, the set $\{l \in \mathbb{R} \mid T_S^{l-}(\omega) < T_S^{l+}(\omega)\}$ is countable due to the monotonicity of $T_S^l(\omega)$ in l. In conjunction with our estimates (3.32) and (3.33), this yields the identities

$$I = II = \mathbb{E}\left[\left(Y_S^{l^*} - Y_S^{l_*}\right) 1_A\right] . \tag{3.34}$$

Moreover, monotonicity of T_S^{l-}, T_S^l, and T_S^{l+} in conjunction with $T_S^{l-} \leq T_S^l \leq T_S^{l+}$ and $\partial_l f \leq 0$ implies

$$I \geq \mathbb{E}\left[\int_{l_*}^{l^*} \int_S^{T_S^l} \partial_l f(t, l)\, dt\, dl\, 1_A\right] \geq II .$$

Together with (3.34), the preceding inequality finally implies

$$\mathbb{E}\left[\left(Y_S^{l^*} - Y_S^{l_*}\right) 1_A\right] = \mathbb{E}\left[\int_{l_*}^{l^*} \int_S^{T_S^l} \partial_l f(t,l)\, dt\, dl\, 1_A\right].$$

As $A \in \mathcal{F}_S$ is arbitrary, this completes the proof of assertion (iii). □

3.4.2 Proof of Existence

Using the notation from the preceding section, let us now define

$$L(\omega,t) \triangleq \sup\{l \in \mathbb{R} \mid Y^l(\omega,t) = X(\omega,t)\} \quad \text{for} \quad (\omega,t) \in \Omega \times [0,\hat{T}) \tag{3.35}$$

and let us put $L(\omega,\hat{T}) \triangleq -\infty$ for $\omega \in \Omega$.

Proposition 3.15 *The process L defined by* (3.35) *is optional and takes values in* $[-\infty,+\infty)$ *almost surely. For every $S \in \mathcal{S}$, each of the following events is contained in the next:*

$$\begin{aligned} A &\triangleq \{(\omega,t,l) \mid l > \sup_{S(\omega)\le v\le t} L(\omega,v)\} \\ \subset B &\triangleq \{(\omega,t,l) \mid T_S^l(\omega) \ge t\} \\ \subset C &\triangleq \{(\omega,t,l) \mid l \ge \sup_{S(\omega)\le v< t} L(\omega,v)\}, \end{aligned}$$

and we have $A = B = C$ up to a $\mathbb{P}\otimes dt \otimes dl$–null set.

PROOF :

1. The process L is optional since, for every $l \in \mathbb{R}$, we have

$$\{(\omega,t) \in \Omega \times [0,\hat{T}] \mid L(\omega,t) > l\} = \bigcup_{l<r\in\mathbb{Q}} \{Y^r = X\}$$

where the latter set is optional by optionality of Y^r and X. To see that L takes values in $[-\infty,+\infty)$, consider $S \in \hat{\mathcal{S}}$ and note that on $\{L(S) = +\infty\}$ we have $X(S) = Y^l(S)$ for all $l \in \mathbb{R}$ almost surely. This entails, in particular, that on $\{L(S) = +\infty\}$ we have

$$X(S) \le \mathbb{E}\left[X(\hat{T}) \,\middle|\, \mathcal{F}_S\right] + \int_S^{\hat{T}} f(t,l)\, dt$$

for all $l \in \mathbb{R}$ almost surely. Letting $l \uparrow +\infty$, we see that this implies

$$\{L(S) = +\infty\} \subset \{X(S) = -\infty\}$$

up to a $\mathbb{P}$–null set. The right event has probability zero by assumption on X and, thus, also $\mathbb{P}[L(S) = +\infty] = 0$.

2. The claimed inclusions $A \subset B \subset C$ are easily derived from the definitions of L and T_S^l. Moreover, we have

$$C \backslash A = \{(\omega, t, l) \mid l = L(\omega, t) \text{ or } l = \sup_{S(\omega) \le v < t} L(\omega, v) \text{ or } t = S(\omega)\}$$

which obviously is a $\mathbb{P} \otimes dt \otimes dl$–null set.

$\square$

We now can give the

Proof of Theorem 3.4 Fix a stopping time $S \in \mathcal{S}$ with $S < \hat{T}$ $\mathbb{P}$–a.s.

1. We first prove the relation

$$X(S) = \mathbb{E}\left[X(T_S^l) \,\middle|\, \mathcal{F}_S\right] + \mathbb{E}\left[\int_S^{T_S^l} f(t, \sup_{S \le v \le t} L(v))\,dt \,\middle|\, \mathcal{F}_S\right] \quad \text{on} \quad \{L(S) \le l\}$$

for every $l \in \mathbb{R}$.

Fix $l_0 \in \mathbb{R}$. The definition of $L(S)$ and absolute continuity of $l \mapsto Y^l(\omega, S(\omega))$ for $\mathbb{P}$–a.e. $\omega \in \Omega$ imply

$$X(S) = Y^{L(S)}(S) = Y^{l_0}(S) - \int_{L(S)}^{l_0} \partial_l Y^l(S)\,dl$$

almost surely on $\{L(S) \le l_0\}$. Due to our formula (3.28) for the density $(\partial_l Y^l(S),\ l \in \mathbb{R})$, the last expression is

$$= Y^{l_0}(S) - \mathbb{E}\left[\int_{L(S)}^{l_0} \int_S^{T_S^l} \partial_l f(t,l)\,dt\,dl \,\middle|\, \mathcal{F}_S\right].$$

Let I denote the above conditional expectation. Fubini's theorem yields

$$I = \mathbb{E}\left[\int_S^{\hat{T}} \int_{L(S)}^{l_0} \partial_l f(t,l) 1_{\{T_S^l \ge t\}}\,dl\,dt \,\middle|\, \mathcal{F}_S\right] \quad \text{on} \quad \{L(S) \le l_0\}.$$

As the sets B and C of Proposition 3.15 coincide up to a $\mathbb{P} \otimes dt \otimes dl$–null set, we may replace the set $\{T_S^l \ge t\}$ in the above expression by $\{l \ge \bar{L}(S,t)\}$ where $\bar{L}(S,t) \stackrel{\Delta}{=} \sup_{S \le v \le t} L(v)$. This yields

$$\begin{aligned} I &= \mathbb{E}\left[\int_S^{\hat{T}} \int_{\bar{L}(S,t) \wedge l_0}^{l_0} \partial_l f(t,l)\,dl\,dt \,\middle|\, \mathcal{F}_S\right] \\ &= \mathbb{E}\left[\int_S^{\hat{T}} \{f(t,l_0) - f(t, \bar{L}(S,t) \wedge l_0)\}\,dt \,\middle|\, \mathcal{F}_S\right] \end{aligned} \tag{3.36}$$

almost surely on $\{L(S) \le l_0\}$.

We claim that

$$f(t,l_0) - f(t,\bar{L}(S,t) \wedge l_0) = \big(f(t,l_0) - f(t,\bar{L}(S,t))\big)\, 1_{\{T_S^{l_0} \ge t\}} \quad dt\text{–a.e.} \tag{3.37}$$

Indeed, the left side of this equality is equal to

$$\begin{aligned}\big(f(t,l_0) - f(t,\bar{L}(S,t))\big)\, 1_{\{l_0 > \bar{L}(S,t)\}} &\ge \big(f(t,l_0) - f(t,\bar{L}(S,t))\big)\, 1_{\{T_S^{l_0} \ge t\}} \\ &\ge \big(f(t,l_0) - f(t,\bar{L}(S,t))\big)\, 1_{\{l_0 \ge \bar{L}(S,t-)\}}\end{aligned}$$

where both estimates are due to the inclusions derived in Proposition 3.15. Since $\bar{L}(S,.)$ is increasing in t, we have $\bar{L}(S,t) = \bar{L}(S,t-)$ Lebesgue–a.e. t and, therefore, the last term in the preceding estimate coincides with the first term dt–a.e. This proves our claim (3.37).

Claim (3.37) in conjunction with (3.36) gives us

$$I = \mathbb{E}\left[\int_S^{T_S^{l_0}} \{f(t,l_0) - f(t,\bar{L}(S,t))\}\, dt \,\middle|\, \mathcal{F}_S\right] \quad \text{on} \quad \{L(S) \le l_0\}\,.$$

Now, resuming our initial calculation, we see that this representation and optimality of $T_S^{l_0}$ imply

$$\begin{aligned}X(S) &= Y^{l_0}(S) - I \\ &= \mathbb{E}\left[X(T_S^{l_0}) + \int_S^{T_S^{l_0}} f(t,l_0)\, dt \,\middle|\, \mathcal{F}_S\right] - \mathbb{E}\left[\int_S^{T_S^{l_0}} \{f(t,l_0) - f(t,\bar{L}(S,t))\}\, dt \,\middle|\, \mathcal{F}_S\right] \\ &= \mathbb{E}\left[\, X(T_S^{l_0}) \,\middle|\, \mathcal{F}_S\right] + \mathbb{E}\left[\int_S^{T_S^{l_0}} f(t,\bar{L}(S,t))\, dt \,\middle|\, \mathcal{F}_S\right]\end{aligned}$$

on $\{L(S) \le l_0\}$. This is what we wanted to show in this first step.

2. As our second step, we show that we have $T_S^l = \hat{T}$ $\mathbb{P}$–a.s. for l sufficiently large.

   Indeed, by continuity and monotonicity of f, there is a constant l^* such that $f(t,l) < 0$ for all $t \in [0,\hat{T}]$ and all $l \ge l^*$. Moreover, we have $X(T) \ge 0$ for every stopping time $T \in \mathcal{S}(S)$ by assumption. For $l \ge l^*$ this implies

$$\mathbb{E}\left[X(T) + \int_S^T f(t,l)\, dt \,\middle|\, \mathcal{F}_S\right] \ge \mathbb{E}\left[0 + \int_S^{\hat{T}} f(t,l)\, dt \,\middle|\, \mathcal{F}_S\right]$$

   for every $T \in \mathcal{S}(S)$. Hence, $\hat{T}$ is an optimal stopping time. In fact, it is the only optimal stopping time because the above estimate is strict if $\mathbb{P}[T < \hat{T}] > 0$. Thus, $T_S^l = \hat{T}$ for $l \ge l^*$.

3. Now, combine the results of Steps 1 and 2 to deduce

$$X(S) = \mathbb{E}\left[\int_S^{\hat{T}} f(t, \sup_{S \le v \le t} L(v))\, dt \,\middle|\, \mathcal{F}_S\right] \quad \text{on} \quad \{L(S) \le l\}$$

for every $l \ge l^*$. Letting $l \uparrow +\infty$, we deduce

$$X(S) = \mathbb{E}\left[\int_S^{\hat{T}} f(t, \sup_{S \le v \le t} L(v))\, dt \,\middle|\, \mathcal{F}_S\right] \quad \text{on} \quad \{L(S) < +\infty\}\,.$$

As the latter event has probability one by Proposition 3.15, this shows that indeed L solves the representation problem (3.1).

□

Existence of a solution to the representation problem (3.1) probably can be proved also under weaker conditions than those of Theorem 3.4. Ideally, one should be able to prove a characterization of representable processes similar to our Theorems 3.2 and 3.3 in the case of certainty.

Furthermore, it should be possible to dispense with the nonnegativity assumption $X \ge 0$. In fact, nonnegativity of X is only used in Step 2 of the proof of this theorem. This step can be replaced, e.g., by an argument showing that $T_S^l \uparrow \hat{T}$ and $\mathbb{E}\left[X(T_S^l) \,\middle|\, \mathcal{F}_S\right] \to 0$ as $l \uparrow +\infty$.

Also differentiability of $f = f(t, l)$ in l might not be necessary, as is suggested by our treatment of the deterministic case where this assumption is not needed. Removing this condition, however, seems to be more involved as the Envelope Theorem (Lemma 3.14 (iii)) forms a crucial part of our argument. One possibility to remove this condition could consist in developing a theory of stochastic convex envelopes, analogous to our approach in the case of certainty. This would rely on an appropriate definition of stochastic convexity, a task we relegate to future work.

Chapter 4

Explicit Solutions

This chapter provides explicit solutions to the Hindy–Huang–Kreps utility maximization problem in a complete financial market. The key to these solutions is the method described at the end of Chapter 2. We first calculate explicitly the minimal level of satisfaction for every Lagrange multiplier. Within the induced family of optimal consumption plans, we then determine the unique policy which exhausts the investor's budget.

We carry out this program first for the case of certainty when prices for consumption decline at a constant exponential interest rate. It turns out that, if the investor is not 'too impatient', the minimal level equation can be solved explicitly using our previously introduced concept of inhomogeneously convex envelopes. The key insight for this explicit description of the minimal level is the observation that a sufficiently small rate of time–preference guarantees local inhomogeneous convexity of the exponential price density. This observation allows an easy computation of the corresponding envelopes. It then only remains to determine the first and the last time of consumption, and this is achieved by the principle of smooth fit.

Theorem 4.1 yields the economically intuitive result that an investor with a low initial level of satisfaction immediately starts consuming by taking an initial gulp, whereas a high initial level of satisfaction induces him to wait for a while. After that consumption occurs at rates until from some time strictly before the investor's time horizon on, the investor refrains from consuming for the rest of his lifetime. This behavior is rational for a Hindy–Huang–Kreps utility maximizer since — in contrast to his time–additive counterpart — he obtains utility from past consumption rather than from current consumption alone. For the special case of a time–homogeneous, power–felicity function, we can dispense with the aforementioned assumption of a sufficiently patient investor. In fact, Theorem 4.2 describes the complete closed–form solution to the utility maximization problem for all possible model paramters, extending the result in Hindy, Huang, and Kreps (1992). Such an explicit description of optimal plans under certainty has also been obtained in Bank and Riedel (2000). Note, however, that the argument there consists in verifying the Kuhn–Tucker characterization of optimal plans directly, while our method is based on the minimal level of satisfaction. In fact, the minimal level approach seems more convenient since it is also applicable in a more general setting where, e.g., interest rates are not constant.

The greater flexibility of the minimal level approach is also illustrated by our solution under uncertainty in a homogeneous setting where the methods of classic calculus used in Bank and Riedel (2000) are no longer applicable. We consider a setting with infinite time horizon in which the investor's preferences are given by a power–felicity function. Consumption prices follow a geometric Lévy process with finite Laplace exponent. In this setting, we can determine the minimal level of satisfaction explicitly by an easy calculation. Theorem 4.3 describes the induced optimal plans. Using the Wiener–Hopf factorization, we characterize in Theorem 4.4 those parameter values for which the problem is well–posed. For a large class of state–price driving Lévy processes, results from fluctuation theory allow us to describe explicitly the dual relation between Lagrange multipliers and different amounts of initial wealth. This also enables us to compute the investor's indirect utility in closed–form.

We finally carry out several case studies illustrating the flexibility of both the Hindy–Huang–Kreps framework for intertemporal consumption choice and of our minimal level method. It turns out that, depending on the kind of underlying stochastics, a whole variety of optimal consumption plans can occur. If, for instance, the state–price driven by Brownian motion, then optimal plans turn out to be singular. If there is a downward price jump, one can observe consumption in gulps. Even consumption at rates is possible, e.g., if prices follow a geometric Poisson process with upward jumps.

4.1 The Case of Certainty

In this section, we solve explicitly a deterministic version of the utility maximization problem studied in Chapter 2. More precisely, we consider an investor with Hindy–Huang–Kreps utility functional U as in Section 1.2.2 who wishes to spend his initial wealth $w \geq 0$ for consumption over the period $[0, \hat{T}]$. As in the Arrow–Debreu framework, we assume as given a complete set of forward markets, where the consumption good is traded at the deterministic price

$$\psi(t) \stackrel{\Delta}{=} e^{-\int_0^t r(s)\,ds} \; (0 \leq t \leq \hat{T}) \tag{4.1}$$

where $r : [0, \hat{T}] \to \mathbb{R}$ is some Lebesgue–integrable deterministic interest rate process. The agent buys his preferred consumption plan at time 0. He, thus, faces the optimization problem to find the *deterministic* plan with maximal utility in his budget–feasible set:

$$\text{Maximize } U(C) \text{ over } C \in \mathcal{M}_+ \text{ subject to } \Psi(C) \stackrel{\Delta}{=} (\psi, C) \leq w\,. \tag{4.2}$$

4.1.1 The Deterministic Minimal Level Equation

In order to solve problem (4.2) explicitly, we will follow the minimal level approach described at the end of Chapter 2. Here, we will use our results on the deterministic version of the minimal level equation studied in Section 3.3 of the preceding chapter.

Our starting point is the following corollary to Theorem 3.2.

Corollary 4.1 *Assume that U is a Hindy–Huang–Kreps utility functional with felicity function u satisfying Assumption 1.2, and let the price density ψ be given by* (4.1)*.*

Then the minimal level process for problem (4.2) *exists. More precisely, for every Lagrange multiplier $M > 0$, there is a unique upper–semicontinuous function $L = L^M : [0, \hat{T}) \to \mathbb{R}_+$ such that*

$$M\psi(s) = \int_s^{\hat{T}} \partial_y u \left(t, \sup_{s \le v \le t} \left\{ L(v) e^{B(v)-B(t)} \right\} \right) \beta(s) e^{B(s)-B(t)} \, dt$$

for all $s \in [0, \hat{T})$, where $B(t) \triangleq \int_0^t \beta(s) \, ds$.

This function L has the representation

$$L(s) = i \left(s, -e^{B(s)} \partial^+ \breve{x}^s(s) \right) \quad (0 \le s < \hat{T}) \,. \tag{4.3}$$

Here, $i(s, .) \triangleq \partial_y u(s, .)^{-1}$ is the inverse of marginal felicity; x is the function

$$x(t) \triangleq \frac{M\psi(t) e^{-B(t)}}{\beta(t)} 1_{[0,\hat{T})}(t) \quad (0 \le t \le \hat{T}) \,, \tag{4.4}$$

and $\breve{x}^s$ denotes the $(-f)$–convex envelope of the restriction $x|_{[s,\hat{T}]}$ $(s \in [0, \hat{T}))$ where

$$f(t,l) \triangleq \begin{cases} \partial_y u(t, -e^{-B(t)}/l) e^{-B(t)} & \text{for } 0 \le t \le \hat{T} \text{ and } l < 0, \\ -l & l \ge 0 \,. \end{cases} \tag{4.5}$$

Proof : Let x and f be defined by (4.4) and (4.5), respectively. The Inada–condition $\partial_y u(t, +\infty) = 0$ ensures continuity of f. In conjunction with strict concavity of $u(t, .)$, the other Inada–condition $\partial_y u(t, +0) = +\infty$ guarantees that $f(t, .)$ is strictly decreasing from $+\infty$ to $-\infty$. Thus, the function f satisfies our Assumption 3.1. Moreover, x is continuous on $[0, \hat{T})$ with a downward jump in $t = \hat{T}$ and, thus, lower–semicontinuous.

Hence, we can apply Theorem 3.2 to obtain existence of a unique upper–semicontinuous function $\tilde{l} : [0, \hat{T}) \to \mathbb{R}$ such that

$$x(s) = \int_s^{\hat{T}} f(t, \sup_{s \le v \le t} \tilde{l}(v)) \, dt \quad \text{for all} \quad s \in [0, \hat{T}] \,. \tag{4.6}$$

From this result and from our Uniqueness Theorem 3.1, we infer that $\tilde{l}(s)$ $(0 \le s < \hat{T})$ has the characterizations

$$-\partial^+ \breve{x}^s(s) = f(s, \tilde{l}(s)) \quad \text{and} \quad \tilde{l}(s) = \inf_{s<t\le\hat{T}} \tilde{l}_{s,t} \tag{4.7}$$

where $\tilde{l}_{s,t}$ is defined by

$$x(s) - x(t) = \int_s^t f(u, \tilde{l}_{s,t})\, du \quad (t \in (s, \hat{T}])\,.$$

From the latter representation, we deduce that

$$\tilde{l}(s) \le \tilde{l}_{s,\hat{T}} < 0 \quad \text{for all} \quad s \in [0, \hat{T})\,,$$

where $\tilde{l}_{s,\hat{T}} < 0$ holds true since $x(s) - x(\hat{T}) = x(s) > 0$ on $[0, \hat{T})$. Hence $\tilde{l}$ being negative, we may rewrite (4.6) in the form

$$M\psi(s) = \int_s^{\hat{T}} \partial_y u(t, -e^{-B(t)}/\sup_{s\le v\le t} \tilde{l}(v))\beta(s)e^{B(s)-B(t)}\, dt\,.$$

Defining $L(v) \stackrel{\Delta}{=} -\, e^{-B(v)}/\tilde{l}(v)$ $(0 \le v < \hat{T})$, we thus obtain an upper–semicontinuous function L satisfying the asserted minimal level equation. Its characterization (4.3) is an immediate consequence of the analogous characterization (4.7) for $\tilde{l}$. □

4.1.2 Explicit Construction of the Minimal Level

Due to the preceding result, we can determine the minimal level of satisfaction by computing the initial derivative of every $(-f)$–convex envelope $\breve{x}^t$ $(t \in [0, \hat{T}))$, where x and f are given by (4.4) and (4.5) respectively. As we shall see, this can be done explicitly under the following two assumptions.

Assumption 4.1 *The problem is time–homogeneous in the sense that*

(i) the Hindy–Huang–Kreps agent's level of satisfaction decays with constant rate $\beta(t) \equiv \beta > 0$*, i.e., for every consumption plan* $C \in \mathcal{M}_+$*, the functional* $Y(C)$ *takes the form*

$$Y(C)(t) \stackrel{\Delta}{=} \eta e^{-\beta t} + \beta \int_0^t e^{-\beta(t-s)}\, dC(s) \quad (0 \le t \le \hat{T})$$

for some constants $\eta \ge 0$*,* $\beta > 0$*;*

(ii) the interest rate is constant: $r(t) \equiv r$*.*

Assumption 4.2 *The investor's felicity function u satisfies Assumption 1.2. In addition, it has continuous partial derivatives $\partial_y u$, $\partial_y^2 u$, and $\partial_t \partial_y u$, all defined on $[0, \hat{T}] \times (0, +\infty)$ such that*

$$\mathcal{L}u(t,y) > 0 \quad \textit{for all} \quad 0 \le t \le \hat{T},\ y > 0\,.$$

where $\mathcal{L}$ denotes the differential operator $\mathcal{L} \stackrel{\Delta}{=} r\partial_y - \beta y \partial_y^2 + \partial_t \partial_y$.

Remark 4.2 *In terms of rate of time preference $\delta \stackrel{\Delta}{=} -\partial_t \partial_y u(t,y)/\partial_y u(t,y)$, interest rate r, relative risk aversion $a \stackrel{\Delta}{=} -y\partial_y^2 u(t,y)/\partial_y u(t,y)$ and rate of decay β, the condition $\mathcal{L}u(t,y) > 0$ is equivalent to $\delta < r + \beta a$. Hence, we assume that the investor's rate of time preference is sufficiently small as compared to interest rate, risk aversion and rate of decay.*

The following condition ensures that, also when the interest rate is negative, wealth does not decrease faster than the agent's satisfaction.

Assumption 4.3 $r + \beta > 0$.

Lemma 4.3 *Under Assumption 4.1, it is optimal to consume the whole wealth in a single initial gulp whenever Assumption 4.3 is violated.*

PROOF : We directly verify that under the given conditions the plan $C^* \equiv w$ satisfies the first–order conditions for optimality. In the considered special case, this amounts to verifying that

$$\frac{\nabla U(C^*)}{\psi}(s) = e^{(r+\beta)s} \int_s^{\hat{T}} \partial_y u(t, (\eta + \beta w)e^{-\beta t})\beta e^{-\beta t}\, dt$$

is decreasing. Indeed, we have

$$\partial_s \frac{\nabla U(C^*)}{\psi}(s) = (r+\beta)\frac{\nabla U(C^*)}{\psi}(s) - \partial_y u(s, (\eta + \beta w)e^{-\beta s})\beta e^{rs}\,,$$

and this is ≤ 0 for $r + \beta \le 0$ since both ∇U and $\partial_y u > 0$ are nonnegative by assumption on u. □

In case Assumptions (4.1)–(4.3) are satisfied, the key observation is that the function x of Corollary 4.1 is locally inhomogeneously convex:

Lemma 4.4 *Under Assumptions 4.1–4.3, each restriction $x|_{[0,T]}$ $(T \in [0, \hat{T}))$ is strictly $(-f)$–convex.*

PROOF : Put $g \triangleq -f$ and let $G(t,.)$ denote the inverse $G(t,.) \triangleq g(t,.)^{-1}$. In order to prove g–convexity of any $x|_{[0,T]}$ $(T \in [0,\hat{T}))$, we verify the characterization of g–convexity given in Proposition 3.8 (iii). Thus, we show that, on $[0,\hat{T})$, the derivative

$$\dot{x}(t) = -M(r+\beta)e^{-(r+\beta)t}$$

can be written in the form

$$\dot{x}(t) = g(t, a(t))$$

for some strictly increasing function $a : [0,\hat{T}) \to \mathbb{R}$.

Due to Assumption 4.2, we may apply the Implicit Function Theorem to obtain that the unique function a satisfying the above equality is differentiable with derivative

$$\dot{a}(t) = \frac{\partial}{\partial t} G(t, \dot{x}(t)) \, .$$

Straightforward computations yield

$$\dot{a}(t) = \frac{\partial}{\partial t}\left(e^{\beta t} I^M(t)\right) = -e^{\beta t}\frac{\mathcal{L}u}{\partial_y^2 u}(t, I^M(t)) \tag{4.8}$$

where $I^M(t) \triangleq i(t, e^{-rt}M(r+\beta)/\beta)$ with $i(t,.) \triangleq (\partial_y u(t,.))^{-1}$. As $\mathcal{L}u > 0$ and $\partial_y^2 u < 0$ by Assumption 4.2, equation (4.8) shows that indeed a is strictly increasing. □

The explicit construction of the minimal level now follows from Lemma 3.12:

Lemma 4.5 *Under Assumptions 4.1–4.3, the minimal level of satisfaction L^M $(M > 0)$ for the deterministic Hindy–Huang–Kreps utility maximization problem* (4.2) *is given as the continuous function*

$$L^M(t) = \begin{cases} I^M(t) & \text{for } t \in [0, t^*(M))\,, \\ l^M_{t,\hat{T}} e^{-\beta t} & \text{for } t \in [t^*(M), \hat{T}) \end{cases} \tag{4.9}$$

where

(i) I^M is defined by

$$I^M(t) \triangleq i(t, M\tfrac{r+\beta}{\beta}e^{-rt})\,, \tag{4.10}$$

(ii) $l^M_{t,\hat{T}}$ is the unique constant satisfying

$$Me^{-(r+\beta)t} = \int_t^{\hat{T}} \partial_y u(s, e^{-\beta s} l^M_{t,\hat{T}})\beta e^{-\beta s}\, ds\,, \tag{4.11}$$

and

(iii) $t^*(M)$ is the unique solution $t \in (0,\hat{T})$ to

$$e^{(r+\beta)t}\int_t^{\hat{T}} \partial_y u\left(s, I^M(t)e^{-\beta(s-t)}\right)\beta e^{-\beta s}\, ds = M \tag{4.12}$$

provided there is some, and $t^*(M) = 0$ otherwise.

PROOF : Lemma 4.4 allows us to apply Lemma 3.12 to our situation. This yields existence of a time $t^* = t^*(M) < \hat{T}$ such that the initial derivatives $\partial^+ \breve{x}^t(t)$ $(0 \le t < \hat{T})$ are given by (3.20). In conjunction with representation (4.3) of L^M obtained in Corollary 4.1, this yields the claimed formula (4.9) for the minimal level. Continuity of L^M is also deduced from Lemma 3.12, since x is continuously differentiable on $[0, \hat{T})$.

Hence, it merely remains to prove the claimed characterization (iii) of $t^*(M)$. We first consider the case $t^*(M) = 0$. Here, strict local $(-f)$–convexity of x implies via the last part of Lemma 3.12 that $\partial^+ x(t) > -f(t, l^M_{t,\hat{T}})$ for all $t \in (0, \hat{T}]$. As $\partial^+ x(t) = -f(t, e^{\beta t} I^M(t))$ by definition of I^M, this shows that equation (4.12) cannot have a solution $t \in (0, \hat{T})$ because this would imply $l^M_{t,\hat{T}} = I^M(t)$ by definition of $l^M_{t,\hat{T}}$.

In case $t^*(M) > 0$, the last part of Lemma 3.12 shows that $\partial^+ x(t) = -f(t, l^M_{t,\hat{T}})$ iff $t = t^*(M)$. In light of the identity $\partial^+ x(t) = -f(t, e^{\beta t} I^M(t))$, this means that $t^*(M)$ is the unique solution to equation (4.12). □

The following lemma will be needed to prove continuous dependence of the plan C^M on its Lagrange multiplier $M > 0$.

Lemma 4.6 *The mapping $t^* : (0, +\infty) \to [0, \hat{T})$ defined by*

$$M \mapsto t^*(M) = \begin{cases} \text{the unique solution } t \in (0, \hat{T}) \text{ to } (4.12), \text{ if there is some,} \\ 0, \text{ otherwise} \end{cases}$$

is continuous.

PROOF : Let $\Phi^M(t)$ denote the left side of equation (4.12).

1. Straightforward computations yield

$$\partial_t \Phi^M(t) = (r+\beta)\{\Phi^M(t) - M\} - e^{(r+\beta)t} \int_t^{\hat{T}} \frac{\partial_y^2 u(s, I^M(t) e^{-\beta(s-t)})}{\partial_y^2 u(t, I^M(t))} \mathcal{L}u(t, I^M(t))\, ds$$

   which by Assumption 4.2 is

$$< (r+\beta)\left\{\Phi^M(t) - M\right\}$$

   for $t \in [0, \hat{T})$. This reveals that Φ^M is strictly decreasing at every solution t to equation (4.12).

2. If $M^0 > 0$ is such that $t^*(M^0) > 0$, we have $\Phi^{M^0}(t^*(M^0)) = M^0$ and $\partial_t \Phi^{M^0}(t^*(M^0)) < 0$ by Step 1. Hence, the Implicit Function Theorem shows that equation (4.12) also has a solution $t = t^*(M)$ for M in some open neighborhood of M^0. In addition, it yields that this solution depends continuously on M. This proves the continuity of $t^*(.)$ in such a point M^0.

3. To prove continuity in points $M^0 > 0$ where there is no solution to (4.12) in $(0, \hat{T}]$, consider a sequence $M^n \to M^0 > 0$ $(n \uparrow +\infty)$ with corresponding $t^n \in (0, \hat{T}]$ satisfying (4.12) for $M \triangleq M^n$ and $t \triangleq t^n$. It suffices to prove that necessarily $t^n \to 0$ $(n \uparrow +\infty)$. Suppose to the contrary that this sequence has an accumulation point $t^0 \in (0, T]$. Since the left side of (4.12), $\Phi^M(t)$, is jointly continuous in (t, M), we have
$$\Phi^{M^0}(t^0) = \lim_n \Phi^{M^n}(t^n) = \lim_n M^n = M^0 \,,$$
at least along some suitable subsequence. Thus, $t = t^0 \in (0, \hat{T}]$ is a solution to (4.12) for $M \triangleq M^0$ in contradiction to our assumption on M^0.

□

4.1.3 Optimal Plans

Having determined the minimal level of satisfaction L^M for every Lagrange multiplier $M > 0$, the next step is to describe the plan which tracks this level.

Lemma 4.7 *Let C^M denote the consumption plan tracking the level L^M given by Lemma 4.5. Let furthermore $t^*(M) \in [0, \hat{T})$ be defined as in this lemma and put*
$$t_*(M) \triangleq \inf \left\{ t \in [0, \hat{T}] \mid \eta < L^M(t) e^{\beta t} \right\} \in [0, \hat{T}) \cup \{+\infty\} \,. \tag{4.13}$$
If $t_*(M) = +\infty$, then $C^M = 0$.

Otherwise, $t_*(M)$ is the first time of consumption and $t^*(M) \geq t_*(M)$ is the last time. If $t^*(M) = t_*(M) = 0$ then C^M is the plan which prescribes to take an initial gulp of size $(l^M_{0,\hat{T}}(0) - \eta)^+/\beta$ and not to consume otherwise, i.e.,
$$dC^M(t) = \frac{(l^M_{0,\hat{T}}(0) - \eta)^+}{\beta} \, \delta_0(dt)$$
where δ_0 denotes the Dirac–measure on $[0, \hat{T}]$ with unit mass in $t = 0$.

In case $t^*(M) > 0$, we have
$$dC^M(t) = \frac{(I^M(0) - \eta)^+}{\beta} \, \delta_0(dt) - \frac{\mathcal{L}u}{\beta \partial_y^2 u}(t, I^M(t)) 1_{(t_*(M), t^*(M))}(t) \, dt \,,$$
i.e., the plan C^M prescribes a consumption gulp at time $t = 0$ provided the initial level of satisfaction is less than $I^M(0)$, and it prescribes to consume at rates over the time period $(t_*(M), t^*(M))$.

PROOF : By definition, we have

$$dC^M(t) = \tfrac{1}{\beta} e^{-\beta t}\, dA^M(t)$$

where

$$A^M(0-) \triangleq \eta, \quad A^M(t) \triangleq \eta \vee \sup_{0 \le v \le t} \left\{ e^{\beta v} L^M(v) \right\} \quad (0 \le t \le \hat{T})\,; \tag{4.14}$$

compare Lemma 2.15.

1. We first show that $v \mapsto e^{\beta v} L^M(v)$ is strictly increasing on $[0, t^*(M))$ and strictly decreasing on $[t^*(M), \hat{T})$.

   Indeed, its $[0, t^*(M))$–component, $v \mapsto e^{\beta v} I^M(v)$, is strictly increasing on the hole interval $[0, \hat{T}]$ as can be read off equation (4.8) in the proof of Lemma 4.4. The derivative of its $[t^*(M), \hat{T})$–component, $v \mapsto l^M_{v,\hat{T}}$, satisfies

   $$\frac{\partial}{\partial v} l^M_{v,\hat{T}} = \frac{\partial_y u(v, e^{-\beta v} l^M_{v,\hat{T}}) - M \frac{r+\beta}{\beta} e^{-rv}}{e^{\beta v} \int_v^{\hat{T}} \partial_y^2 u(t, e^{-\beta t} l^M_{v,\hat{T}}) e^{-2\beta t}\, dt}\,.$$

   Hence, $v \mapsto l^M_{v,\hat{T}}$ and, thus, also $v \mapsto e^{\beta v} L^M(v)$ is strictly decreasing on

   $$[t^*(M), \hat{T}) = \left\{ v \in [0, \hat{T}) \;\middle|\; l^M_{v,\hat{T}} < I^M(v) e^{\beta v} \right\}\,.$$

2. From Step 1 and the definition of A^M, one easily deduces

   $$A^M(t) = \eta \vee \left\{ e^{\beta(t \wedge t^*(M))} L^M(t \wedge t^*(M)) \right\} \quad (0 \le t \le \hat{T})\,. \tag{4.15}$$

   Hence, $t_*(M)$ defined by (4.13) is the first point of increase of A^M and, thus, also the first time of consumption provided $t_*(M) < +\infty$. Similarly, $t^*(M)$ is the last point of increase of A^M and, thus, the last time of consumption.

   If $t^*(M) = t_*(M) = 0$ then A^M increases only by a single jump at time 0. This means that the agent only consumes at the beginning (if at all), namely by taking an initial gulp

   $$\Delta C^M(0) = (l^M_{0,\hat{T}} - \eta)^+/\beta\,.$$

   If $t^*(M) > 0$ and $t_*(M) < +\infty$, then A^M increases in a differentiable manner on $(t_*(M), t^*(M))$ which means that the agent consumes at rates

   $$dC^M(t) = \tfrac{1}{\beta} e^{-\beta t} \frac{\partial}{\partial t} \left(I^M(t) e^{\beta t} \right) = -\frac{\mathcal{L}u}{\beta \partial_y^2 u}(t, I^M(t))$$

   over this time interval. In case $t_*(M) = 0 < t^*(M)$, A^M possibly has an initial jump which amounts to an additional initial consumption gulp of size

   $$\Delta C^M(0) = (I^M(0) - \eta)^+/\beta\,.$$

□

The next lemma shows that, among the consumption plans C^M $(M > 0)$ of Lemma 4.7, there is a plan which exhausts the investor's budget.

Lemma 4.8 *The mapping $M \mapsto \Psi(C^M)$ has image $[0, +\infty)$.*

PROOF : We keep the notation from the preceding proof.

By formula (4.15) and Lemma 4.6, $A^M(t)$ depends continuously on M for all t. By the Portemanteau Theorem, this implies weak*–continuity of $M \mapsto C^M \in \mathcal{M}_+$ and, therefore, also continuity of $M \mapsto \Psi(C^M) = \int_0^{\hat{T}} e^{-rt}\, dC^M(t)$.

Moreover, this mapping has image $[0, +\infty)$, since, in addition, we have

$$\Psi(C^M) \geq ((I^M(0) - \eta)^+ \wedge (l^M_{0,\hat{T}} - \eta)^+)/\beta \to +\infty \quad \text{for} \quad M \downarrow 0\,,$$

and

$$\Psi(C^M) \leq C^M(\hat{T}) \leq A^M(\hat{T})/\beta \to 0 \quad \text{for} \quad M \uparrow +\infty\,.$$

□

Thus, for every initial capital $w \geq 0$ there is a Lagrange multiplier $M(w)$ whose associated consumption plan $C^{M(w)}$ costs $\Psi(C^{M(w)}) = w$. We now can give the explicit solution to our optimization problem (4.2).

Theorem 4.1 *Under Assumptions 4.1–4.3, the optimal policy for the deterministic Hindy–Huang–Kreps utility maximization problem* (4.2) *is to follow the unique consumption plan C^M of Lemma 4.7 whose price exhausts the investor's budget.*

PROOF : Existence of a budget–exhausting plan C^M among those described in Lemma 4.7 follows from Lemma 4.8. As this plan tracks the minimal level L^M, its optimality in $\mathcal{A}(\Psi(C^M)) = \mathcal{A}(w)$ is an immediate consequence of our Minimal Level Theorem 2.3. Uniqueness of the optimal plan has already been established before. Hence, for every $w \geq 0$, there is precisely one plan C^M with price $\Psi(C^M) = w$. □

4.1.4 Homogeneous Felicity Functions

Let us now illustrate the above solution for a Hindy–Huang–Kreps utility with a separable power–felicity function

$$u(t, y) \stackrel{\Delta}{=} u_\alpha(t, y) \stackrel{\Delta}{=} \begin{cases} e^{-\delta t}\frac{1}{\alpha}\,(y^\alpha - 1) & \text{if } 0 \neq \alpha < 1, \\ e^{-\delta t}\log y & \text{if } \alpha = 0. \end{cases} \tag{4.16}$$

In this special case, we obtain the following complete solution:

Theorem 4.2 *Under Assumption 4.1, a Hindy–Huang–Kreps agent with power felicity as in* (4.16) *optimally consumes his whole wealth $w > 0$ in one single gulp at time $t = 0$ iff*

$$r + \beta \leq 0 \quad or \quad r + \beta \leq \delta + \alpha\beta \tag{4.17}$$

or the constant

$$\tau^* \triangleq \begin{cases} \hat{T} + \frac{1}{\delta+\alpha\beta} \log\left(1 - \frac{\delta+\alpha\beta}{r+\beta}\right) & \text{if } \delta + \alpha\beta \neq 0 \\ \hat{T} - \frac{1}{r+\beta} & \text{else} \end{cases} \tag{4.18}$$

is nonpositive.

Otherwise, if $w \geq k^*\eta$ with

$$k^* \triangleq \begin{cases} \frac{r+\beta(1-\alpha)-\delta}{\beta(\delta-\alpha r)} \left(1 - e^{-\frac{\delta-\alpha r}{1-\alpha}\tau^*}\right) & \text{if } \delta \neq \alpha r \\ \left(1 - \frac{\delta-r}{\beta(1-\alpha)}\right)\tau^* & \text{else} \end{cases}$$

it is optimal to have an initial consumption gulp of size

$$\Delta C(0) = \frac{w - k^*\eta}{1 + \beta k^*}$$

and to consume at rates

$$dC(t) = \left(1 - \frac{\delta - r}{\beta(1-\alpha)}\right) \frac{\eta + \beta w}{1 + \beta k^*} e^{-\frac{\delta-r}{1-\alpha}t}\, dt$$

afterwards until time $t = \tau^* > 0$.

In case $w < k^*\eta$ the agent optimally waits until time

$$\tau_*(M) = \frac{1}{r + \beta(1-\alpha) - \delta} \log\left(M \frac{r+\beta}{\beta} \eta^{1-\alpha}\right) \quad (M > 0)\,, \tag{4.19}$$

and then he starts consuming at rates

$$dC^M(t) = \left(1 - \frac{\delta - r}{\beta(1-\alpha)}\right) \left(M \frac{r+\beta}{\beta} e^{(\delta-r)t}\right)^{-\frac{1}{1-\alpha}} dt$$

until time $t = \tau^* > 0$. Here, $M = M(w) > 0$ is determined as $M \triangleq \frac{\beta}{r+\beta}K$ where $K > 0$ is the (unique) solution to

$$K^{-\frac{r+\beta}{r+\beta(1-\alpha)-\delta}} \eta^{-\frac{\delta-\alpha r}{r+\beta(1-\alpha)-\delta}} - K^{-\frac{1}{1-\alpha}} e^{-\frac{\delta-\alpha r}{1-\alpha}\tau^*} = \frac{\beta(\delta - \alpha r)}{r + \beta(1-\alpha) - \delta} w$$

if $\delta \neq \alpha r$, and to

$$\eta^{-\frac{1}{1-\alpha}} \left\{ \frac{1}{\beta} \log K + \frac{1-\alpha}{\beta} \log y - (r + \beta(1-\alpha) - \delta)\tau^* \right\} = -(1-\alpha)w$$

in case $\delta = \alpha r$.

Remark 4.9 *Note that the preceding solution is complete in the sense that we only require the natural conditions $\alpha < 1$, $\beta, \hat{T} > 0$, and $\eta \geq 0$; the parameters r, δ can be chosen arbitrarily in $\mathbb{R}$.*

PROOF :

1. We first treat the case where condition (4.17) does *not* hold true, i.e., where

$$r+\beta>0 \quad \text{and} \quad r+\beta>\delta+\alpha\beta\,.$$

The first of these conditions is just Assumption 4.3. The second condition guarantees that the agent's felicity function meets Assumption 4.2. Indeed, u obviously meets Assumption 1.2 and we have

$$\mathcal{L}u(t,y)=(r+\beta(1-\alpha)-\delta)e^{-\delta t}y^{-(1-\alpha)}>0$$

under the second parameter restriction.

Assumptions 4.1–4.3 being satisfied, we now can apply Theorem 4.1 to determine the optimal plan.

We start with the determination of the last time of consumption $t^*(M)$. By Lemma 4.5, $t^*(M)>0$ iff equation (4.12) has a solution $t\in(0,\hat{T})$. In our case, equation (4.12) reads $\Phi^M(t)=M$ where

$$\Phi^M(t)=\begin{cases} M\frac{r+\beta}{\delta+\alpha\beta}\left(1-e^{-(\delta+\alpha\beta)(\hat{T}-t)}\right) & \text{if } \delta+\alpha\beta\neq 0,\\ M(r+\beta)(\hat{T}-t) & \text{else.}\end{cases}$$

As $r+\beta(1-\alpha)-\delta>0$ by assumption, $\Phi^M(t)=M$ has the solution $t=\tau^*\in(-\infty,\hat{T})$. Thus, we have $t^*(M)=\tau^*$ if $\tau^*>0$ and $t^*(M)=0$ if $\tau^*\leq 0$.

The initial time for consumption is also easily computed. We find $t_*(M)=\tau_*(M)^+$ if $\tau_*(M)\in(-\infty,\hat{T}]$ and $t_*(M)=+\infty$ otherwise.

Due to Lemma 4.7, consumption rates on the (possibly empty) time interval $(t_*(M),t^*(M))$ are given by

$$dC^M(t)=\left(1-\frac{\delta-r}{\beta(1-\alpha)}\right)\left(M\frac{r+\beta}{\beta}e^{(\delta-r)t}\right)^{-\frac{1}{1-\alpha}}\,dt\,.$$

According to the parameter values and the value of w, several situations arise. If τ^* is nonpositive, we have $t^*(M)=0$ for all $M>0$; in this case, it is optimal to consume the whole wealth immediately by a single gulp at time $t=0$.

For $\tau^*>0$, we distinguish three cases. First, for $0<M\leq M_*\stackrel{\Delta}{=}\frac{\beta}{r+\beta}\eta^{-(1-\alpha)}$, we have $t_*(M)=0$. Hence, for these values of M consumption starts immediately at time 0, namely with an initial gulp of size

$$\Delta C^M(0)=\frac{1}{\beta}\left((M\tfrac{r+\beta}{\beta})^{-\frac{1}{1-\alpha}}-\eta\right)\,.$$

After that consumption occurs at rates until the last time of consumption $\tau^* > 0$. This allows us to compute the prices for the consumption streams C^M $(0 < M \leq M_*)$ as

$$\Psi(C^M) = \left(M\frac{r+\beta}{\beta}\right)^{-\frac{1}{1-\alpha}} \left\{\frac{1}{\beta} + \frac{r+\beta(1-\alpha)-\delta}{\beta(\delta-\alpha r)}\left(1-e^{-\frac{\delta-\alpha r}{1-\alpha}\tau^*}\right)\right\} - \frac{\eta}{\beta}$$

if $\delta \neq \alpha r$, and as

$$\Psi(C^M) = \left(M\frac{r+\beta}{\beta}\right)^{-\frac{1}{1-\alpha}} \left\{\frac{1}{\beta} + \left(1-\frac{\delta-r}{\beta(1-\alpha)}\right)\tau^*\right\} - \frac{\eta}{\beta}$$

in case $\delta = \alpha r$.

Second, for $M_* < M < M^*$ where

$$M^* \triangleq \frac{\beta}{r+\beta}\eta^{-(1-\alpha)}e^{(r+\beta(1-\alpha)-\delta)\tau^*},$$

the consumption plan C^M has the price

$$\begin{aligned}\Psi(C^M) = \frac{r+\beta(1-\alpha)-\delta}{\beta(\delta-\alpha r)}\Bigg\{&\left(M\frac{r+\beta}{\beta}\right)^{-\frac{r+\beta}{r+\beta(1-\alpha)-\delta}}\eta^{-\frac{\delta-\alpha r}{r+\beta(1-\alpha)-\delta}} \\ &-\left(M\frac{r+\beta}{\beta}\right)^{-\frac{1}{1-\alpha}}e^{-\frac{\delta-\alpha r}{1-\alpha}\tau^*}\Bigg\}\end{aligned}$$

for $\delta \neq \alpha r$, and

$$\Psi(C^M) = \left(1-\frac{\delta-r}{\beta(1-\alpha)}\right)\left(M\frac{r+\beta}{\beta}\right)^{-\frac{1}{1-\alpha}}(\tau^*-\tau_*(M))$$

if $\delta = \alpha r$.

Finally, $M \geq M^*$ implies $t_*(M) = +\infty$. Hence, $C^M = 0$ and $\Psi(C^M) = 0$ for these values of M.

Now, given the investor's initial capital $w \geq 0$, we merely have to find $M(w) > 0$ with $\Psi(C^{M(w)}) = w$ — the corresponding plan $C^{M(w)}$ is then optimal in $\mathcal{A}(w)$. Using the above formulas for $\Psi(C^M)$ $(M > 0)$, this leads to the claimed explicit description of the optimal consumption plan.

2. Let us now consider the case where condition (4.17) does hold true. In this case, we have to show optimality of the 'greedy' consumption plan $C^* \equiv w$ where the agent consumes his whole wealth in a single initial gulp.

To this end, we shall verify directly the first–order conditions given in Theorem 2.2. Clearly, as consumption occurs only at the beginning, this boils down to showing that $\nabla U(C^*)/\psi$ attains its maximum over $[0, \hat{T}]$ at time $t = 0$.

We compute

$$\frac{\nabla U(C^*)}{\psi}(t) = \frac{e^{(r+\beta)t}}{(\eta+\beta w)^{(1-\alpha)}} \cdot \begin{cases} (e^{-(\delta+\alpha\beta)t} - e^{-(\delta+\alpha\beta)\hat{T}})/(\delta+\alpha\beta) & \text{if } \delta+\alpha\beta \neq 0\,, \\ \hat{T} - t & \text{otherwise.} \end{cases}$$

Obviously, the second factor is decreasing in t for any value of $\delta + \alpha\beta$.

If $r + \beta \leq 0$, also the first factor is decreasing in t and, thus, optimality of the greedy plan C^* follows in this case.

Under the complementary parameter restriction $0 < r + \beta \leq \delta + \alpha\beta$, we have, in particular, $\delta + \alpha\beta > 0$ and therefore $\nabla U(C^*)/\psi$ is a positive multiple of

$$t \mapsto e^{((r+\beta)-(\delta+\alpha\beta))t} - e^{(r+\beta)t-(\delta+\alpha\beta)\hat{T}}\,.$$

Obviously, this is a decreasing function under the above parameter restrictions. This establishes optimality of C^* in this case.

□

Remark 4.10 *The independence of the last time of consumption from the investor's initial wealth is due to the homogeneity of the felicity function in this special case. For general felicity functions this independence will no longer hold true.*

Let us now briefly discuss some of the economic implications of the preceding theorem.

First, in our context the investor optimally refrains from consumption from a certain point in time on. From an economic point of view, this may be a surprising feature of the above solution, as it is in contrast to the infinite horizon setting in Hindy, Huang, and Kreps (1992) (henceforth HHK) or to a setup using time–additive utilities and habit–formation as in, e.g., Constantinides (1990), Sundaresan (1989). In fact, the solution in HHK may be recovered from ours when $\hat{T} \uparrow +\infty$, assuming (as in HHK) that $\alpha r < \delta < r + \beta(1-\alpha)$ and $\delta + \alpha\beta > 0$. To illustrate the difference to the time–additive setup, one might consider the case $\delta = r \geq 0$ and initial standard of living $\eta = 0$. Then a time–additive utility maximizer consumes at constant rates $c = (rw)/(1-e^{-r\hat{T}})$. His standard of living thus increases from zero up to the level $c(1-e^{-\beta\hat{T}})$ at time $\hat{T}$. A HHK–utility maximizer takes an initial gulp of size $w/(1+\beta\frac{1-e^{-r\tau^*}}{r})$ to lift his standard of living up to a desired level, and he keeps it on this level afterwards until time $\tau^* < \hat{T}$ when he has spent all his wealth. To do so, a HHK–utility maximizer transfers wealth from the distant future to the present. This behavior is rational because he still obtains utility from past consumption even when he refrains from consuming. Loosely speaking, being old, he enjoys having had a good time as a young man.

A second surprising feature is that, even when interest rates are negative, it may be optimal for the agent not to spend all the money for consumption immediately at the beginning. In fact, this happens when satisfaction decays faster than wealth ($r+\beta > 0$), granted the agent is not too impatient ($\delta < r + \beta(1-\alpha)$).

4.2 The Case of Uncertainty

In this section, we turn to the general utility maximization problem under uncertainty as it is discussed in Chapter 2. Thus, we consider an economic agent with Hindy–Huang–Kreps utility U as in Section 1.2.2 who acts as a price–taker on a complete financial market with optional state–price density ψ. Given initial wealth $w \geq 0$, his problem is to find the optimal budget–feasible consumption plan $C \in \mathcal{C}$:

$$\text{Maximize } V(C) \stackrel{\Delta}{=} \mathbb{E}U(C) \text{ over } C \in \mathcal{C} \text{ subject to } \langle \psi, C \rangle \leq w. \tag{4.20}$$

In a first step, we verify our Assumption 2.4 which guarantees the existence of the minimal level of satisfaction. In a second step, we introduce additional homogeneity assumptions which allow us to derive a closed–form solution of our optimization problem.

4.2.1 Existence of the Minimal Level of Satisfaction

As an analogue of Corollary 4.1 in the case of certainty, let us first provide an existence result for the minimal level of satisfaction in the general stochastic case.

Corollary 4.11 *Suppose that the investor's felicity function u satisfies Assumption 1.2 and that, in addition, it has a second–order partial derivative $\partial_y^2 u \in C([0,\hat{T}] \times (0,+\infty))$ satisfiying the growth condition*

$$-\lim_{y \uparrow +\infty} \partial_y^2 u(t,y) i'(i^{-1}(y)) = d(t) \quad \textit{for every} \quad t \in [0,\hat{T}]\,. \tag{4.21}$$

for some increasing diffeomorphism $i : (-\infty, 0) \to (0,+\infty)$ and some continuous function $d : [0,\hat{T}] \to \mathbb{R}$.

Then Assumption 2.4 is satisfied, i.e., for every Lagrange multiplier $M > 0$, there is a unique nonnegative, progressively measurable process $L = L^M$ with upper–rightcontinuous paths and $L(\hat{T}) = 0$ such that the state–price density ψ allows the representation

$$M\psi(S) = \mathbb{E}\left[\int_S^{\hat{T}} \partial_y u \left(t, \sup_{S \leq v \leq t} \left\{ L(v) e^{B(v)-B(t)} \right\} \right) \beta(S) e^{B(S)-B(t)}\, dt \,\middle|\, \mathcal{F}_S \right]$$

for every stopping time $S \in \hat{\mathcal{S}}$, where $B(t) \triangleq \int_0^t \beta(s)\,ds$.

Moreover, we have the representation

$$L(S) = \operatorname*{ess\,inf}_{T \in \mathcal{S}^>(S)} l_{S,T} \tag{4.22}$$

at every stopping time $S \in \hat{\mathcal{S}}$. Here, we let

$$l_{S,T} \triangleq +\infty \quad on \quad A \triangleq \{\mathbb{E}[\,X(S) - X(T)\,|\,\mathcal{F}_S] \leq 0\}$$

and, on A^c, we define $l_{S,T}$ as the unique $\mathcal{F}_S$–measurable random variable such that

$$\mathbb{E}[\,X(S) - X(T)\,|\,\mathcal{F}_S] = \mathbb{E}\left[\int_S^T \partial_y u(t, l_{S,T}) e^{-B(t)}\,dt \,\middle|\, \mathcal{F}_S\right]\;; \tag{4.23}$$

here, X denotes the process $X(s) \triangleq (M\psi(s)e^{-B(s)}/\beta(s))1_{[0,\hat{T})}(s)$ $(0 \leq s \leq \hat{T})$.

Condition (4.21) is of purely technical importance. It ensures differentiability of the auxiliary function f which will be defined in our proof of the above corollary. Essentially, this condition requires that $\partial_y^2 u(t, y)$ converges to 0 sufficiently fast and uniformly in $t \in [0, \hat{T}]$ as $y \uparrow +\infty$. A sufficient criterion for this condition is given in

Lemma 4.12 *If the felicity function u takes the separable form $u(t, y) = d(t)v(y)$, we can choose the diffeomorphism $i : (-\infty, 0) \to (0, +\infty)$ required for Corollary 4.11 as*

$$i(l) \triangleq (v')^{-1}(-l) \quad (l < 0)\,.$$

PROOF : Strict concavity of u entails that v' is strictly decreasing and, thus, invertible. The Inada–conditions and continuous differentiability of u up to second order ensure that i is indeed an increasing diffeomorphism $(-\infty, 0) \to (0, +\infty)$. Moreover, we have

$$-\partial_y^2 u(t, y) i'(i^{-1}(y)) = d(t) \left.\frac{\partial}{\partial l}\right|_{-l = v'(y)} (v'(i(l))) = d(t)\,.$$

Thus, the limit in (4.21) obviously exists and is $= d(t)$. As u is continuous, so is d, and this completes the proof of our assertion. □

Let us now give the

Proof of Corollary 4.11 Put $X(t) \triangleq (M\psi(t)e^{-B(t)}/\beta(t))1_{[0,\hat{T})}(t)$ and define

$$f(t, l) \triangleq \begin{cases} \partial_y u(t, e^{-B(t)} i(l)) e^{-B(t)} & \text{for } l < 0, \\ -d(t)l - l^2 & \text{for } l \geq 0, \end{cases} \quad (0 \leq t \leq \hat{T})$$

where i and d are as in the formulation of the corollary. Due to the Inada–condition $\partial_y u(t, +\infty) = 0$ and continuity of d, the function f is a continuous mapping $[0, \hat{T}] \times \mathbb{R} \to \mathbb{R}$; strict concavity of $u(t, .)$ and the other Inada–condition $\partial_y u(t, +0) = +\infty$ ensure that $f(t, .)$ is strictly decreasing from $+\infty$ to $-\infty$. Moreover, condition (4.21) implies that $f(t, .)$ is continuously differentiable. Thus, the function f defined above satisfies all assumptions needed for Theorem 3.4.

Also the process X satisfies the assumptions required in this theorem. Indeed, with Z denoting the density process for $\mathbb{P}^* \approx \mathbb{P}$, we may write X as the product of the class (D)–supermartingale $Z 1_{[0,\hat{T})}$ and the continuous, bounded process

$$\frac{\exp\left(-\int_0^t \{r(s) + \beta(s)\}\, ds\right)}{\beta(t)} \quad (0 \le t \le \hat{T})\,.$$

This shows that $X \ge 0$ is dominated by a martingale and lower–semicontinuous in expectation.

Having verified its assumptions, we now can apply our Theorem 3.4 to obtain existence of an optional process $\tilde{L}$ such that

$$X(S) = \mathbb{E}\left[\int_S^{\hat{T}} f(t, \sup_{S \le v \le t} \tilde{L}(v))\, dt \,\middle|\, \mathcal{F}_S \right] \tag{4.24}$$

for every stopping time $S \in \mathcal{S}$. More precisely, we infer from the proof of this theorem that $\tilde{L}$ can be chosen as

$$\tilde{L}(t) \stackrel{\Delta}{=} \sup\{l \in \mathbb{R} \mid Y^l(t) = X(t)\} \quad (0 \le t < \hat{T}), \quad \tilde{L}(\hat{T}) \stackrel{\Delta}{=} -\infty,$$

where Y^l is as in Lemma 3.14. Obviously,

$$Y^0(S) = \operatorname*{ess\,inf}_{T \in \mathcal{S}(S)} \mathbb{E}\left[X(T) \mid \mathcal{F}_S \right] \le \mathbb{E}\left[X(\hat{T}) \,\middle|\, \mathcal{F}_S \right] = 0 < X(S)$$

for every stopping time $S \in \hat{\mathcal{S}}$. Hence, $Y^0 \le 0$ on $[0, \hat{T})$ and therefore

$$0 \ge \sup\{l \in \mathbb{R} \mid Y^l(t) = X(t)\} = \tilde{L}(t)$$

for all $t \in [0, \hat{T}]$, i.e., $\tilde{L} \le 0$. This allows us to rewrite formula (4.24) as

$$X(S) = \mathbb{E}\left[\int_S^{\hat{T}} \partial_y u \left(t, e^{-B(t)} i(\sup_{S \le v \le t} \tilde{L}(v)) \right) e^{-B(t)}\, dt \,\middle|\, \mathcal{F}_S \right] \quad (S \in \mathcal{S})\,.$$

By Lemma 3.1, we may pass to the upper–rightcontinuous modification $\tilde{L}'$ of $\tilde{L}$ to obtain another progressively measurable solution of the above representation problem. It now is easy to see that

$$L(\omega, v) \stackrel{\Delta}{=} e^{-B(v)} i(\tilde{L}'(\omega, v)) \quad (v \in [0, \hat{T}))\,, \qquad L(\omega, \hat{T}) \stackrel{\Delta}{=} 0 \quad (\omega \in \Omega)$$

is the desired progressively measurable process that solves the minimal level equation.

Finally, the claimed representation (4.22) of $L(S)$ $(S \in \hat{\mathcal{S}})$ follows from the analogous representation of $\tilde{L}'$ given in our Uniqueness Theorem 3.1. □

Some Heuristics

In order to determine the minimal level of satisfaction explicitly, we now could try to solve the optimal stopping problem (4.22) for every stopping time $S \in \hat{\mathcal{S}}$. This, however, is a tedious task in general as already indicated in Remark 3.2. Instead, let us try to find a plausible candidate for the minimal level of satisfaction, and to verify directly that this candidate in fact solves the minimal level equation under appropriate conditions.

To this end, we first recall the structure of optimal consumption plans as they are derived in the 'classical' theory based on time–additive von Neumann–Morgenstern utility functionals. In such a setting, utility is obtained from the current *rate* of consumption, rather than from the instantaneous level of satisfaction. Applying methods of convex duality (confer, e.g., Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987)), one shows that the marginal felicity of an optimal consumption rate for this problem should equal some fixed multiple of the state–price density. This leads to the absolutely continuous optimal consumption plan $dC^{ac}(t) \equiv i(t, K\psi(t))\, dt$, where $i(t,.) \stackrel{\Delta}{=} (\partial_y u(t,.))^{-1}$ is the inverse of marginal felicity and K is a strictly positive constant.

At least formally, the level of satisfaction $Y(C)$ plays the same role for our utility functional $U(C)$ as does the rate of consumption for the classic von Neumann–Morgenstern utilities. Thus, the above solution suggests to choose $C \in \mathcal{C}$ such that $Y(C)(t) \equiv i(t, K\psi(t))$. However, the right side of this equality will typically be of unbounded variation, while the left side has bounded variation for any choice of $C \in \mathcal{C}$. Hence, there might be no $C \in \mathcal{C}$ inducing a level of satisfaction of the form suggested above. But we can try to stay as close as possible to this desirable level. This suggest to choose the consumption plan $C^K \stackrel{\Delta}{=} C^{L^K}$ which tracks the level process

$$L^K(t) \stackrel{\Delta}{=} i(t, K\psi(t)) \quad (t \geq 0)\,. \tag{4.25}$$

Any process L^K $(K > 0)$ gives us a plausible candidate for the minimal level of satisfaction L we are looking for. In fact, this turns out to be the right choice in a homogeneous setting as we shall see in the next section.

4.2.2 The Minimal Level in a Homogeneous Setting

In order to verify, that the candidate derived in the preceding section indeed solves the minimal level equation, we will impose two homogeneity assumptions.

The first of these assumptions refers to the form of uncertainty the investor faces.

Assumption 4.4 *The state–price density ψ is homogeneous in the sense that*

$$\psi(t) = \exp\left(-\theta X(t) - (r + \pi(-\theta))t\right) \quad (t \geq 0),$$

for constants $\theta > 0$, $r \in \mathbb{R}$ and some $(\mathbb{P}, \mathbb{F})$–Lévy process X with finite Laplace–exponent $\pi(\xi)$ for all $\xi \in \mathbb{R}$.

Hence, interest rates are constant, $r(t) \equiv r$, and uncertainty is introduced by a stochastic process X with stationary and independent increments which possesses all exponential moments

$$\mathbb{E}\exp\left(\xi X(t)\right) < +\infty \quad (\xi \in \mathbb{R},\ t \geq 0)\,.$$

The Laplace–exponent $\pi(.)$ of such a process X is defined via

$$\mathbb{E}\exp\left(\xi X(t)\right) = \exp(\pi(\xi)t) \quad \text{for all } \xi \in \mathbb{R},\ t \geq 0\,;$$

see, e.g., Bertoin (1996). The constant $\theta > 0$ can be viewed as the 'market price of risk'.

Example 4.13 *1. For $X = (W(t),\ t \geq 0)$, a standard Brownian motion, we have $\pi(\xi) = \frac{1}{2}\xi^2$, and the state–price density*

$$\psi(t) = \exp\left(-\theta W(t) - \left(r + \tfrac{1}{2}\theta^2\right)t\right) \quad (t \geq 0)$$

takes the well–known form of a geometric Brownian motion. This specification of ψ corresponds to the setup studied in Hindy and Huang (1993).

2. If $X = (\pm N(t),\ t \geq 0)$ is a Poisson process with upward (downward) jumps and intensity λ, then $\pi(\xi) = \lambda(e^{\pm\xi} - 1)$ and, therefore,

$$\psi(t) = \exp\left(\mp\theta N(t) - \left(r + \lambda(e^{\mp\theta} - 1)\right)t\right) \quad (t \geq 0)$$

is a geometric Poisson process.

Remark 4.14 *Note that the above examples describe complete financial markets if $\mathbb{F}$ is the augmented filtration generated by X.*

The second assumption essentially means that the agent's preferences are homogeneous.

Assumption 4.5 *The agent has homogeneous Hindy–Huang–Kreps utility in the sense that*

(i) his time horizon is infinite: $\hat{T} = +\infty$;

(ii) he has a power felicity function

$$u(t, y) = e^{-\delta t}\frac{1}{\alpha}y^{\alpha} \quad (t \geq 0,\ y > 0)$$

for some constant $\alpha \in (-\infty, 1)\backslash\{0\}$*;*

(iii) his satisfaction decays at the constant rate $\beta(t) \equiv \beta$*, i.e., for every consumption plan* $C \in \mathcal{C}$*, his satisfaction evolves according to*

$$Y(C)(t) \stackrel{\Delta}{=} \eta e^{-\beta t} + \beta \int_0^t e^{-\beta(t-s)}\, dC(s) \quad (t \geq 0)$$

for some constants $\eta, \beta > 0$*.*

Remark 4.15 *(i) The case* $\alpha = 0$*, corresponding to '*$\log$*–felicity', can be treated with the same method as the 'power–felicities' above. For ease of exposition, we leave this case to the interested reader.*

(ii) Due to the infinite time–horizon $\hat{T} = +\infty$*, the above setting does not fit exactly into the general framework of the preceding chapters. However, because either* $u < 0$ *or* $u \geq 0$*, both*

$$U(C) \stackrel{\Delta}{=} \int_0^{+\infty} u(t, Y(C)(t))\, dt = \int_0^{+\infty} e^{-\delta t}\frac{1}{\alpha}(Y(C)(t))^{\alpha}\, dt$$

and $V(C) = \mathbb{E}U(C)$ *are still well–defined as functionals taking possibly infinite values. Also the gradients*

$$\nabla U(C)(t) \stackrel{\Delta}{=} \int_t^{+\infty} \partial_y u(s, Y(C)(s))\beta e^{-\beta(s-t)}\, ds \quad (0 \leq t \leq \hat{T})$$

and $\nabla V(C) = {}^{o}\nabla U(C)$ *exist as processes taking values in* $[0, +\infty]$*. Hence, our general Sufficiency Lemma 2.7 and, thus, also the Minimal Level Theorem 2.3 is still applicable in this setting.*

To ensure that $V(0)$ is finite, we have to make

Assumption 4.6 $\delta + \alpha\beta > 0$.

For $\alpha \in (0, 1)$ this condition is also necessary (not sufficient, see Theorem 4.4 below) to ensure that the problem is well–posed since otherwise $V \equiv +\infty$.

The preceding assumptions allow us to determine explicitly the solution to our minimal level equation.

Lemma 4.16 *Under Assumptions 4.4–4.6, our heuristically derived candidate*

$$L^K(t) \stackrel{\Delta}{=} \left(Ke^{\delta t}\psi(t)\right)^{-\frac{1}{1-\alpha}} \quad (t \geq 0)$$

of equation (4.25) *is in fact the minimal level of satisfaction for Lagrange multiplier*

$$M \stackrel{\Delta}{=} \mathbb{E}\left[\int_0^{+\infty} \beta e^{-(\delta+\alpha\beta)s} \inf_{0\leq v\leq t}\{e^{-(\beta(1-\alpha)-\delta)v}\psi(v)\}\, dt\right] K < +\infty\,. \tag{4.26}$$

PROOF : For any stopping time S, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_S^{+\infty} \partial_y u(t, e^{-\beta t} \sup_{S\leq v\leq t}\{L^K(v)e^{\beta v}\})\beta e^{-\beta(t-S)}\, dt \,\middle|\, \mathcal{F}_S\right] \\
&= \mathbb{E}\left[\int_S^{+\infty} \beta e^{\beta S} e^{-(\delta+\alpha\beta)t} \inf_{S\leq v\leq t}\{Ke^{-(\beta(1-\alpha)-\delta)v}\psi(v)\}\, dt \,\middle|\, \mathcal{F}_S\right] \\
&= \mathbb{E}\left[\int_0^{+\infty} \beta e^{-(\delta+\alpha\beta)t} \inf_{0\leq v\leq t}\left\{Ke^{-(\beta(1-\alpha)-\delta)v}\frac{\psi(S+v)}{\psi(S)}\right\} dt \,\middle|\, \mathcal{F}_S\right] \psi(S) \\
&= \mathbb{E}\left[\int_0^{+\infty} \beta e^{-(\delta+\alpha\beta)t} \inf_{0\leq v\leq t}\{Ke^{-(\beta(1-\alpha)-\delta)v}\psi(v)\}\, dt\right] \psi(S)
\end{aligned}$$

where the last equation holds true because X is a $(\mathbb{P}, \mathbb{F})$–Lévy process. Thus, L^K does indeed solve our minimal level equation (2.18) for $M = M(K) > 0$ as defined in (4.26). Note that $M < +\infty$ because the infimum in its definition is always less than or equal to 1 and because $\delta + \alpha\beta > 0$ by Assumption 4.6. □

4.2.3 Optimal Plans

From Lemma 2.15 we easily infer that the consumption plan $C^K \stackrel{\Delta}{=} C^{L^K}$ which tracks the level $L = L^K$ of Lemma 4.16 can be represented in the following form:

$$dC^K(t) = \frac{1}{\beta} e^{-\beta t}\, dA^K(t) \quad (t \geq 0) \tag{4.27}$$

where, for $t \geq 0$,

$$A^K(0-) \stackrel{\Delta}{=} \eta, \quad A^K(t) \stackrel{\Delta}{=} \eta \vee \left\{K^{-\frac{1}{1-\alpha}} \exp\left(\frac{S(t)}{1-\alpha}\right)\right\} \tag{4.28}$$

with S denoting the running supremum

$$S(t) \stackrel{\Delta}{=} \sup_{0\leq v\leq t} X_{\theta,\mu}(v)$$

of the affine transformation

$$X_{\theta,\mu}(t) \stackrel{\Delta}{=} \theta X(t) + \mu t \tag{4.29}$$

of X with 'volatility' θ and 'drift'

$$\mu \stackrel{\Delta}{=} \pi(-\theta) + r + \beta(1-\alpha) - \delta\,. \tag{4.30}$$

As before, we have

$$Y(C^K)(t) = e^{-\beta t} A^K(t) \quad (t \geq 0)\,.$$

Figure 4.1 illustrates this definition, in case that the driving process X is a Brownian motion. If X is a Poisson process with upward and downward jumps, the consumption

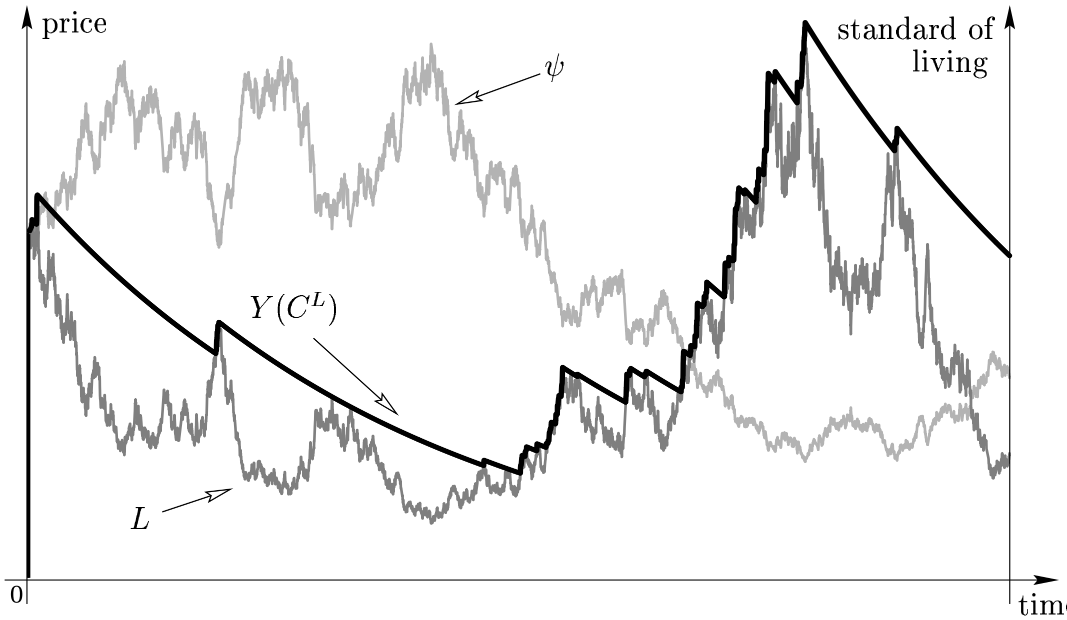

Figure 4.1: Typical paths for the state–price ψ (light grey line), the level of satisfaction $Y(C^L)$ (black line), and its minimal level L (grey line) in a Brownian setting.

plans look considerably different; see Figure 4.2.

Tracking the minimal level of satisfaction is optimal also in this setting:

Theorem 4.3 *Under Assumptions 4.4–4.6, the consumption plan C^K of* (4.27) *is optimal for the Hindy–Huang–Kreps utility maximization problem* (4.20).

PROOF : The same argument as in the proof of Theorem 2.3 shows that C^K satisfies the first–order conditions for Lagrange multiplier $M \in (0, +\infty)$ with (4.26) also in the present infinite time horizon case $\hat{T} = +\infty$. The claim thus follows from our general Sufficiency Lemma 2.5. □

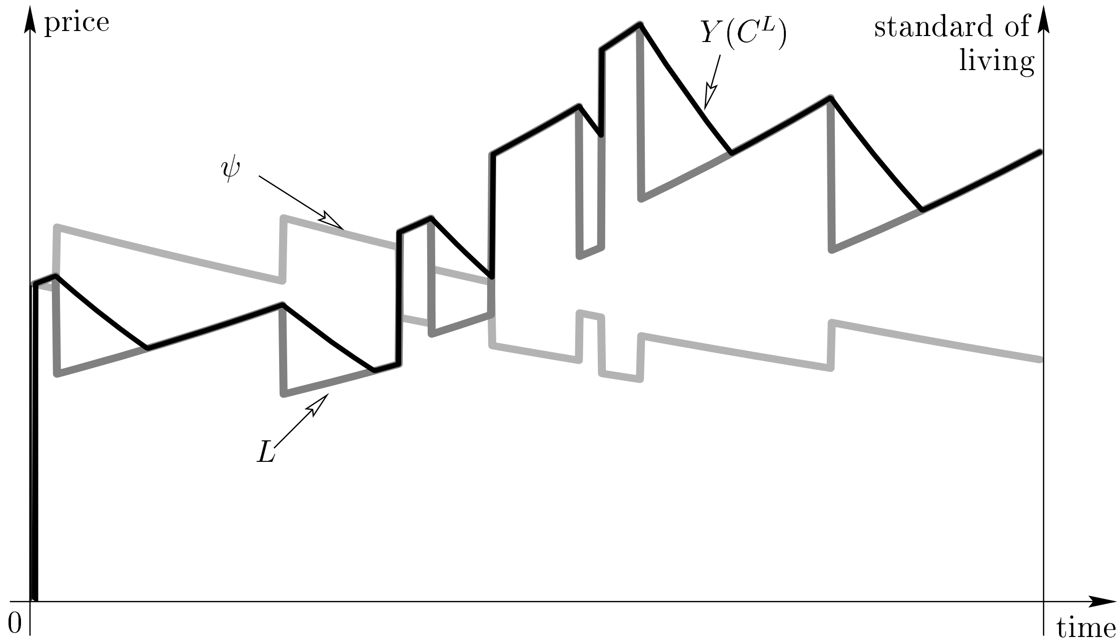

Figure 4.2: Typical paths for the state–price ψ (light grey line), the level of satisfaction $Y(C^L)$ (black line), and its minimal level L (grey line) in a Poissonian setting.

4.2.4 Prices and Utilities

The preceding section shows that the consumption plans C^K $(K > 0)$ are optimal in their respective class. However, it leaves open whether their price and utility is finite or not. Thus, we still have to check for which parameter values of the problem this condition is satisfied. Furthermore, we have to calculate the exact prices for varying Lagrange multiplier $K > 0$ in order to find the plan whose price coincides with a given initial capital $w > 0$.

Well–Posedness of the Utility Maximization Problem

We show that, in our homogeneous framework, the optimization problem (4.20) is well–posed if and only if all prices of our candidate policies C^K $(K > 0)$ are finite. This means, in particular, that our method yields the complete solution to the utility maximization problem provided it is well–posed. To prove this result, we make the additional

Assumption 4.7 *$r + \beta > 0$.*

Recall that this condition appeared as Assumption 4.3 also in the case of certainty.

Theorem 4.4 *Let Assumptions 4.4–4.7 hold true. Then we have equivalence between*

(i) Finite prices: $\langle \psi, C^K \rangle < +\infty$ for some (all) $K > 0$.

and

(ii) The investor's rate of time preference δ satisfies

$$\delta > \hat{\delta} \triangleq \alpha r + (1-\alpha)\pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \alpha\pi(-\theta)\,. \tag{4.31}$$

For $\alpha < 0$, these two assertions are always satisfied, and the utility–maximization problem is always well–posed. For $\alpha \in (0,1)$, assertions (i) and (ii) are equivalent to

(iii) Finite utilities: $V(C^K) < +\infty$ for some (all) $K > 0$,

and the (joint) violation of these conditions entails that, for any initial wealth $w > 0$, there is a budget–feasible plan C with infinite expected utility $V(C) = +\infty$, i.e., the utility maximization problem is ill–posed.

Remark 4.17 *Note that there is a slight gap in Theorem 4.4, since it leaves open whether or not the optimization problem is well–posed in case $\alpha > 0$ and $\delta = \hat{\delta}$. In Proposition 4.23 below, this case is treated under some additional assumption.*

The proof of Theorem 4.4 will be prepared by the following Lemmata 4.18–4.20.

Lemma 4.18 *(i) In terms of the increasing process A^K, we may express the price of the consumption plan C^K as*

$$\Psi^K \triangleq \left\langle \psi, C^K \right\rangle = \tfrac{1}{\beta}\left(\mathbb{E}^* A^K(\tau^*) - \eta\right) \quad (K > 0)\,, \tag{4.32}$$

where τ^* is an independent exponential random time with parameter $r + \beta > 0$.

(ii) We have $\Psi^K < +\infty$ for some $K > 0$ iff

$$\mathbb{E}^* \exp\left(\frac{S(\tau^*)}{1-\alpha}\right) < +\infty\,, \tag{4.33}$$

where τ^* is as in (i). In particular, the price of every *policy C^K $(K > 0)$ is finite if just one of these prices is finite.*

(iii) The mapping $K \mapsto \Psi^K$ is nonnegative, nondecreasing, and convex. If prices are finite, we have $\Psi^K \uparrow +\infty$ as $K \downarrow 0$ and $\Psi^K \downarrow 0$ as $K \uparrow +\infty$. In particular, for every initial capital $w > 0$ there is a consumption policy C^K with price $\Psi^K = w$ in this case.

Proof : From $dC^K(t) = \frac{1}{\beta} e^{-\beta t}\, dA^K(t)$ and partial integration of the price functional, we deduce that for all $K > 0$

$$\begin{aligned}
\Psi^K &= \mathbb{E}^* \int_0^{+\infty} e^{-rt}\, dC^K(t) \\
&= \frac{1}{\beta} \mathbb{E}^* \lim_{T \uparrow +\infty} \left(A^K(T) e^{-(r+\beta)T} - \eta + \int_0^T A^K(t)(r+\beta) e^{-(r+\beta)t}\, dt \right) . \qquad (4.34)
\end{aligned}$$

Hence,

$$\mathbb{E}^* A^K(\tau^*) = \mathbb{E}^* \int_0^{+\infty} A^K(t)(r+\beta) e^{-(r+\beta)t}\, dt < +\infty \qquad (4.35)$$

is necessary for $\Psi^K < +\infty$. It is also sufficient since it implies

$$\lim_{T \uparrow +\infty} A^K(T) e^{-(r+\beta)T} = 0 \quad \mathbb{P}^*\text{–a.s.} \qquad (4.36)$$

Indeed, otherwise we have $\limsup_{T \uparrow +\infty} A^K(T) e^{-(r+\beta)T} > 0$ with positive $\mathbb{P}^*$–probability. Thus, on a set with positive $\mathbb{P}^*$–measure, there is a random $\varepsilon > 0$ such that

$$A^K(\sigma_n) e^{-(r+\beta)\sigma_n} \geq \varepsilon$$

along a sequence of random times σ_n tending to $+\infty$ as $n \uparrow +\infty$. Without loss of generality we may assume that $\sigma_{n+1} - \sigma_n \geq 1$ for all n. Since A^K is nondecreasing we have $A^K(t) e^{-(r+\beta)t} \geq \varepsilon e^{-(r+\beta)} > 0$ whenever $t \in [\sigma_n, \sigma_n + 1)$ for some n. This implies $\int_0^{+\infty} A^K(t)(r+\beta) e^{-(r+\beta)t}\, dt = +\infty$ with positive $\mathbb{P}^*$–probability. Hence, (4.35) implies (4.36). Furthermore the preceding considerations yield that (i) is implied by (4.34).

For assertion (ii) it remains to note that $\mathbb{E}^* A^K(\tau^*) < +\infty$ is equivalent to $\mathbb{E}^* \exp\left(S(\tau^*)/(1-\alpha)\right) < +\infty$. This follows from

$$K^{-\frac{1}{1-\alpha}} \exp\left(\frac{S(\tau^*)}{1-\alpha} \right) \leq A^K(\tau^*) \leq \eta + K^{-\frac{1}{1-\alpha}} \exp\left(\frac{S(\tau^*)}{1-\alpha} \right) .$$

From (i) we deduce that $K \mapsto \Psi^K$ is nonnegative, nondecreasing, and convex, since so is $K \mapsto A^K$. If prices are finite, $A^{K_0}(\tau^*)$ is $\mathbb{P}^*$–integrable. Thus, $\Psi^K \downarrow 0$ for $K \uparrow +\infty$ by dominated convergence. For $K \downarrow 0$, we have $\Psi^K \geq \Delta C^K(0) \uparrow +\infty$. This yields (iii). □

The following is an analogue of Lemma 4.18 for utilities instead of prices:

Lemma 4.19 *(i) In terms of the increasing process A^K, we may express the expected utility of plan C^K as*

$$V(C^K) = \frac{1}{\alpha(\delta + \alpha\beta)} \mathbb{E} \left(A^K(\tau) \right)^\alpha \quad (K > 0) , \qquad (4.37)$$

where τ is an independent exponential random time with parameter $\delta + \alpha\beta$.

(ii) In case $\alpha \in (0,1)$, we have $V(C^K) < +\infty$ for some $K > 0$ iff

$$\mathbb{E}\exp\left(\frac{\alpha S(\tau)}{1-\alpha}\right) < +\infty\,, \tag{4.38}$$

where τ is as in (i). In particular, the expected utility of every *policy C^K is finite if just one of these utilities is finite.*

PROOF : Note first that, because of Assumption 4.6, we have $\delta+\alpha\beta > 0$, and, therefore, τ is well–defined. Now, (i) follows from $Y(C^K)(t) = e^{-\beta t}A^K(t)$ and the definition of the utility functional $U(.)$. For (ii) we note that, in case $\alpha \in (0,1)$,

$$K^{-\frac{\alpha}{1-\alpha}}\exp\left(\frac{\alpha S(\tau)}{1-\alpha}\right) \le A^K(\tau)^\alpha \le \eta^\alpha + K^{-\frac{\alpha}{1-\alpha}}\exp\left(\frac{\alpha S(\tau)}{1-\alpha}\right)\,.$$

□

Lemma 4.18 and Lemma 4.19 are valid for any semimartingale state–price density which induces a constant interest rate. For the following lemma we need the special Lévy–structure of ψ.

Lemma 4.20 *Let σ be an exponential random time independent of X.*

(i) We have the Wiener–Hopf factorization

$$\mathbb{E}\exp\left(\xi \sup_{0\le s\le\sigma} X(s)\right)\mathbb{E}\exp\left(\xi \inf_{0\le s\le\sigma} X(s)\right) = \mathbb{E}\exp\left(\xi X(\sigma)\right) \tag{4.39}$$

for all $\xi \in \mathbb{R}$.

(ii) If X has no positive jumps and is neither a deterministic drift nor the negative of a subordinator, then $\sup_{0\le s\le\sigma} X(s)$ *is exponentially distributed. The parameter ζ of its distribution is uniquely determined by $\pi(\zeta) = \xi$, where ξ is the parameter of the exponential distribution of σ.*

(iii) Under the risk–neutral measure $\mathbb{P}^*$ induced by ψ, X is again a Lévy process with finite exponential moments. Its $\mathbb{P}^*$–Laplace exponent is given by

$$\pi^*(\xi) = \pi(\xi-\theta) - \pi(-\theta) \quad (\xi \in \mathbb{R})\,. \tag{4.40}$$

PROOF :

(i) For $t \geq 0$, let $\tilde{X}(t) \stackrel{\Delta}{=} \sup_{0 \leq s \leq t} X(s)$. By Theorem VI.5 (i) in Bertoin (1996), the random variables $\tilde{X}(\sigma)$ and $\tilde{X}(\sigma) - X(\sigma)$ are independent. Hence,

$$\begin{aligned}\mathbb{E}\exp(\xi X(\sigma)) &= \mathbb{E}\left[\exp(\xi\tilde{X}(\sigma))\exp(-\xi(\tilde{X}(\sigma) - X(\sigma)))\right] \\ &= \mathbb{E}\exp(\xi\tilde{X}(\sigma))\mathbb{E}\exp(-\xi(\tilde{X}(\sigma) - X(\sigma))) . \qquad (4.41)\end{aligned}$$

Using the Duality Lemma II.2 in Bertoin (1996) and the independence of X and σ, we see that

$$\tilde{X}(\sigma) - X(\sigma) = \sup_{0 \leq s \leq \sigma} \{X((\sigma - s)-) - X(\sigma)\}$$

has the same law as

$$\sup_{0 \leq s \leq \sigma} \{-X(s)\} = -\inf_{0 \leq s \leq \sigma} X(s) .$$

In connection with equation (4.41), this yields (i).

(ii) This is Corollary VII.1.2 in Bertoin (1996).

(iii) By definition of ψ, the density process Z for $\mathbb{P}$ and $\mathbb{P}^*$ is given by

$$Z(t) \stackrel{\Delta}{=} \left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp(-\theta X(t) - \pi(-\theta)t) \quad (t \geq 0) .$$

Hence, for $s, t \geq 0$, we may calculate the conditional $\mathbb{P}^*$–Laplace transform of the increment $X(t+s) - X(t)$ given $\mathcal{F}_t$ as follows:

$$\begin{aligned}&\mathbb{E}^* \left[\exp(\xi(X(t+s) - X(t))) \,|\, \mathcal{F}_t\right] \\ &= \frac{1}{Z(t)}\mathbb{E}\left[\exp(\xi(X(t+s) - X(t)))Z(t+s) \,|\, \mathcal{F}_t\right] \\ &= \frac{1}{Z(t)}\mathbb{E}\left[\exp((\xi - \theta)(X(t+s) - X(t))) \,|\, \mathcal{F}_t\right]\exp(-\theta X(t) - \pi(-\theta)(t+s)) \\ &= \exp(s(\pi(\xi - \theta) - \pi(-\theta))) .\end{aligned}$$

Since the last quantity is deterministic and does not depend on t, the above calculation shows that, also under $\mathbb{P}^*$, the process X has independent and stationary increments. Furthermore, we can easily read off the transformation rule (4.40) for the $\mathbb{P}^*$–Laplace exponent $\pi^*(.)$.

□

Now, we are in a position to give the

Proof of Theorem 4.4

1. We first prove equivalence between (i) and (ii).

   By Lemma 4.18 (ii), we know that (i) is equivalent to

$$\mathbb{E}^* \exp\left(\frac{S(\tau^*)}{1-\alpha}\right) = \mathbb{E}^* \exp\left(\frac{1}{1-\alpha} \sup_{0\le v\le\tau^*} X_{\theta,\mu}(v)\right) < +\infty$$

   where τ^* is an independent exponential random time with parameter $r+\beta > 0$. In turn, the Wiener–Hopf factorization of Lemma 4.20 (i) entails equivalence of this condition and

$$\mathbb{E}^* \exp\left(\frac{X_{\theta,\mu}(\tau^*)}{1-\alpha}\right) = \mathbb{E}^* \exp\left(\frac{\theta X(\tau^*) + \mu\tau^*}{1-\alpha}\right) < +\infty\,.$$

   Since τ^* is independent of X and exponentially distributed with parameter $r+\beta$, we may use Fubini's theorem to obtain equivalence of (i) and

$$r+\beta > \pi^*\left(\frac{\theta}{1-\alpha}\right) + \frac{\mu}{1-\alpha}\,.$$

   Using the transformation rule (4.40), it is finally easy to see that this condition is indeed equivalent to (ii).

2. We next prove *(ii)* $\Leftrightarrow$ *(iii)* for $\alpha \in (0,1)$.

   For these values of α, we may use Lemma 4.19 (ii) and follow a similar line of arguments as in Step 1. This yields that (iii) is equivalent to

$$\mathbb{E}\exp\left(\frac{\alpha}{1-\alpha}\{\theta X(\tau) + \mu\tau\}\right) < +\infty$$

   where τ is an independent exponential random time with parameter $\delta + \alpha\beta > 0$. Using Fubini's theorem allows us to conclude the equivalence of (ii) and

$$\delta + \alpha\beta > \pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \frac{\alpha\mu}{1-\alpha},$$

   which, by an easy calculation, can be shown to be equivalent to (ii).

3. We now verify that (ii) holds true when $\alpha < 0$.

   Indeed, convexity of the Laplace exponent $\pi(.)$ and $\alpha < 0$ imply

$$(1-\alpha)\pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \alpha\pi(-\theta) \le \pi(0) = 0\,.$$

   This implies $\hat{\delta} \le \alpha r$. In turn, Assumptions 4.6 and 4.7 imply $\alpha r < -\alpha\beta < \delta$ for $\alpha < 0$. Thus, (ii) is satisfied.

4. To prove that problem (4.20) is ill–posed for $\alpha \in (0,1)$ in case $\delta < \hat{\delta}$, consider the consumption plan $\bar{C}^K$ obtained from tracking the level process

$$\bar{L}^K(t) \triangleq \left(K e^{\bar{\delta} t} \psi(t)\right)^{-\frac{1}{1-\alpha}} \quad (t \geq 0)$$

where $\bar{\delta} > \hat{\delta}$ is some constant. The corresponding increasing process $\bar{A}^K$ is given by

$$\bar{A}^K(t) = \eta \vee K^{-\frac{1}{1-\alpha}} \exp\left(\frac{1}{1-\alpha} \sup_{0 \leq v \leq t} X_{\theta,\bar{\mu}}(v)\right) \quad (t \geq 0)$$

where

$$\bar{\mu} \triangleq \pi(-\theta) + r + \beta(1-\alpha) - \bar{\delta}\,.$$

From $(i) \Leftrightarrow (iii)$ we know that the price of every policy $\bar{C}^K$ is finite because $\bar{\delta} > \hat{\delta}$. Hence, for any initial wealth $w > 0$, we can find $K = K(w)$ such that $\bar{C}^{K(w)}$ is budget–feasible.

By the same arguments as in the proof of Lemma 4.19, one can now show that the expected utility of the plan $\bar{C}^{K(w)}$ is finite iff

$$\mathbb{E} \exp\left(\frac{\alpha}{1-\alpha} \sup_{0 \leq v \leq \tau} X_{\theta,\bar{\mu}}(v)\right) < +\infty$$

where τ is, as before, an independent exponential random time with parameter $\delta + \alpha\beta > 0$. From the Wiener–Hopf factorization (4.39), we deduce that the above relation holds true iff

$$\mathbb{E} \exp\left(\frac{X_{\theta,\bar{\mu}}(\tau)}{1-\alpha}\right) = \mathbb{E} \exp\left(\frac{\alpha}{1-\alpha}\{\theta X(\tau) + \bar{\mu}\tau\}\right) < +\infty\,.$$

Since τ is independent of X and exponentially distributed, this is equivalent to

$$\delta + \alpha\beta > \pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \frac{\alpha\bar{\mu}}{1-\alpha}\,. \tag{4.42}$$

Now, note that, for $\bar{\delta} \downarrow \hat{\delta}$, the right side of this inequality increases to

$$\pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \frac{\alpha\hat{\mu}}{1-\alpha} = \hat{\delta} + \alpha\beta > \delta + \alpha\beta\,, \tag{4.43}$$

where

$$\hat{\mu} \triangleq \lim_{\bar{\delta} \downarrow \hat{\delta}} \bar{\mu} = \pi(-\theta) + r + \beta(1-\alpha) - \hat{\delta}\,.$$

The equation in (4.43) follows by definition of $\hat{\delta}$. Hence, there are $\bar{\delta} > \hat{\delta}$ for which inequality (4.42) is violated and for which, therefore, the associated plans $\bar{C}^K$ have infinite expected utility, even though their price is finite.

□

Remark 4.21 *As an alternative to Step 3 in the preceding proof, one can refer directly to our general Remarks 2.4 and 2.5 on existence and uniqueness of optimal plans in the infinite horizon case, or one can use the following more abstract argument: Assumption 1.2 implies $V(0) > -\infty$, and $\alpha < 0$ yields $V(C) \leq 0$ for all $C \in \mathcal{C}$. By the remark following our general Sufficiency Lemma 2.7, this already yields that the plans C^K have finite prices, since they satisfy the first–order conditions.*

Some Explicit Computations

In order to obtain closed–form solutions to the optimization problem (4.20), it still remains to calculate all prices $\Psi^K = \langle \psi, C^K \rangle$ $(K > 0)$ and to identify the parameter $K(w)$ for which $\Psi^{K(w)} = w$. This can be done explicitly in the following two cases:

Assumption 4.8 *Either*

(i) $X_{\theta,\mu} = (\theta X(t) + \mu t,\ t \geq 0)$ *with* μ *as in* (4.30) *is a decreasing process.*

or

(ii) $X_{\theta,\mu}$ *is neither a decreasing nor a deterministic process, and all its jumps are nonpositive* ($\Delta X_{\theta,\mu} \leq 0$)*.*

Remark 4.22 *(i) Recall that a Lévy process is decreasing iff it is the negative of a subordinator.*

(ii) The process $X_{\theta,\mu}$ is deterministic iff the prices for consumption are deterministic. This case has already been treated in Section 4.1.

Let τ and τ^* be exponential random times, independent of X with parameter $\delta + \alpha\beta$ and $r + \beta > 0$, respectively. Then, Assumption 4.8 ensures that the suprema $S(\tau)$ and $S(\tau^*)$ are exponentially distributed under $\mathbb{P}$ and $\mathbb{P}^*$ respectively. In fact, if Assumption 4.8 (i) holds true, we evidently have $S \equiv 0$ which corresponds to the parameter values $\zeta = \zeta^* = 0$. Under Assumption 4.8 (ii), we may apply Lemma 4.20 (ii) to obtain the respective exponential parameters $\theta\zeta$, $\theta\zeta^*$ where ζ and ζ^* are the unique positive solutions to

$$\pi(\theta\zeta) + \mu\zeta = \delta + \alpha\beta \quad \text{and} \quad \pi^*(\theta\zeta^*) + \mu\zeta^* = r + \beta \tag{4.44}$$

respectively.

Thus, proceeding from equations (4.32) and (4.37), we now can compute

$$\Psi(C^K) = \frac{1}{\beta} \cdot \begin{cases} \left(K^{-\frac{1}{1-\alpha}} - \eta\right)^+ & \text{if } \zeta^* = 0\,, \\ \frac{(1-\alpha)\zeta^*}{(1-\alpha)\zeta^*-1} K^{-\frac{1}{1-\alpha}} - \eta & \text{if } \eta \leq K^{-\frac{1}{1-\alpha}},\ \zeta^* > 0\,, \\ \frac{1}{(1-\alpha)\zeta^*-1} \eta^{1-(1-\alpha)\zeta^*} K^{-\zeta^*} & \text{else}\,, \end{cases} \tag{4.45}$$

and

$$V(C^K) = \frac{1}{\alpha(\delta+\alpha\beta)} \cdot \begin{cases} \eta^\alpha \vee K^{-\frac{\alpha}{1-\alpha}} & \text{if } \zeta = 0\,, \\ \frac{(1-\alpha)\zeta}{(1-\alpha)\zeta-\alpha} K^{-\frac{\alpha}{1-\alpha}} & \text{if } \eta \leq K^{-\frac{1}{1-\alpha}},\ \zeta > 0\,, \\ \eta^\alpha + \frac{\alpha}{(1-\alpha)\zeta-\alpha}\eta^{\alpha-(1-\alpha)\zeta}K^{-\zeta} & \text{else.} \end{cases}$$

Hence, an agent with initial wealth $w > 0$ optimally follows the consumption plan $C^{K(w)}$ with

$$K(w) \triangleq \begin{cases} (\beta w+\eta)^{-(1-\alpha)} & \text{if } \zeta^* = 0\,, \\ \left(\frac{(1-\alpha)\zeta^*-1}{(1-\alpha)\zeta^*}(\beta w+\eta)\right)^{-(1-\alpha)} & \text{if } w \geq \hat{w},\ \zeta^* > 0\,, \\ \left(((1-\alpha)\zeta^*-1)\eta^{(1-\alpha)\zeta^*-1}\beta w\right)^{-\frac{1}{\zeta^*}} & \text{else}\,, \end{cases}$$

where $\hat{w} \triangleq \frac{1}{\beta}\frac{\eta}{(1-\alpha)\zeta^*-1}$.

Furthermore, using Lemma 4.20 (iii), one can show that $\zeta^* = \zeta + 1$ by a straightforward calculation. This allows us to represent the agent's maximal utility (the value $v(w)$ of the program (4.20)) by

$$v(w) = \frac{1}{\alpha(\delta+\alpha\beta)} \cdot \begin{cases} (\beta w+\eta)^\alpha & \text{if } \zeta^* = 0\,, \\ \zeta\left(\frac{1-\alpha}{(1-\alpha)\zeta-\alpha}\right)^{1-\alpha}\left(\frac{\beta w+\eta}{\zeta+1}\right)^\alpha & \text{if } w \geq \hat{w},\ \zeta^* > 0\,, \\ \eta^\alpha + \alpha\eta^{-\frac{(1-\alpha)\zeta-\alpha}{\zeta+1}}\left(\frac{\beta w}{(1-\alpha)\zeta-\alpha}\right)^{\frac{\zeta}{\zeta+1}} & \text{else.} \end{cases}$$

The above formulae give us the desired explicit solution to the investor's utility maximization problem (4.20) in the homogeneous setting of Assumptions 4.4–4.8.

As pointed out in Remark 4.17, Theorem 4.4 does not characterize completely the parameter values for which problem (4.20) is well–posed in case $\alpha \in (0,1)$. Under Assumption 4.8, this gap can be closed:

Proposition 4.23 *Under Assumption 4.8, the parameter restriction $\delta > \hat{\delta}$ of Theorem 4.4 (iii) is also necessary for problem (4.20) to be well–posed if $\alpha \in (0,1)$. More precisely, suppose that Assumption 4.8 is satisfied and that the parameters of the problem are such that*

$$\delta \leq \hat{\delta} = \alpha r + (1-\alpha)\pi\left(\frac{\alpha\theta}{1-\alpha}\right) + \alpha\pi(-\theta)\,, \quad \alpha \in (0,1)\,. \tag{4.46}$$

Then we have

$$\sup_{C\in\mathcal{A}(w)} V(C) = +\infty$$

for any initial capital $w > 0$.

PROOF : As in the proof of Theorem 4.4, choose some $\bar{\delta} > \hat{\delta}$ and consider, for every $K > 0$, the lower bound $\bar{L}^K$ obtained from L^K by replacing δ with $\bar{\delta}$. Again, the corresponding consumption plans will be denoted by $\bar{C}^K$, and we will write $\bar{S}$ for the analogue of the supremum process S. For simplicity, we assume that $\eta = 0$.

We have

$$\Psi(\bar{C}^K) = \frac{K^{-\frac{1}{1-\alpha}}}{\beta}\mathbb{E}^* \exp\left(\frac{\bar{S}(\tau^*)}{1-\alpha}\right)$$

and

$$V(\bar{C}^K) \geq \mathbb{E}\int_0^\infty e^{-\hat{\delta}t}\frac{1}{\alpha}\left(e^{-\beta t}K^{-\frac{\alpha}{1-\alpha}}e^{\frac{\bar{S}(t)}{1-\alpha}}\right)^\alpha dt = \frac{K^{-\frac{1}{1-\alpha}}}{\alpha(\hat{\delta}+\alpha\beta)}\mathbb{E}\exp\left(\frac{\alpha\bar{S}(\tau)}{1-\alpha}\right)$$

where τ^* and τ are independent exponential random times with parameters $r+\beta > 0$ and $\hat{\delta}+\alpha\beta > 0$, respectively.

In order to meet the budget–constraint, we choose $K > 0$ such that $\Psi(\bar{C}^K) = w$. Note that this is indeed possible because of $\bar{\delta} > \hat{\delta}$. By the above calculations, this gives us

$$v(w) \stackrel{\Delta}{=} \sup_{C\in\mathcal{A}(w)} V(C) \geq V(\bar{C^K}) \geq \frac{(\beta w)^\alpha}{\alpha(\hat{\delta}+\alpha\beta)}\frac{\mathbb{E}\exp\left(\frac{\alpha\bar{S}(\tau)}{1-\alpha}\right)}{\left\{\mathbb{E}^*\exp\left(\frac{\bar{S}(\tau^*)}{1-\alpha}\right)\right\}^\alpha}$$

for all $\bar{\delta} > \hat{\delta}$. Hence, to verify $v(w) \equiv +\infty$, it suffices to prove

$$\frac{\mathbb{E}\exp\left(\frac{\alpha\bar{S}(\tau)}{1-\alpha}\right)}{\left\{\mathbb{E}^*\exp\left(\frac{\bar{S}(\tau^*)}{1-\alpha}\right)\right\}^\alpha} \to +\infty \quad \text{as} \quad \bar{\delta}\downarrow\hat{\delta}\,. \tag{4.47}$$

Using Lemma 4.20 (ii), it is easy to see that

$$\mathbb{E}\exp\left(\frac{\alpha\bar{S}(\tau)}{1-\alpha}\right) = \frac{\bar{\zeta}}{\frac{\alpha}{1-\alpha}-\bar{\zeta}} \quad \text{and} \quad \mathbb{E}^*\exp\left(\frac{\bar{S}(\tau^*)}{1-\alpha}\right) = \frac{\bar{\zeta}^*}{\frac{1}{1-\alpha}-\bar{\zeta}^*}$$

where $\bar{\zeta}$ and $\bar{\zeta}^*$ are determined by

$$\pi(\theta\bar{\zeta}) + \bar{\mu}\bar{\zeta} = \hat{\delta}+\alpha\beta \quad \text{and} \quad \pi^*(\theta\bar{\zeta}^*) + \bar{\mu}\bar{\zeta}^* = r+\beta \tag{4.48}$$

with $\bar{\mu} \stackrel{\Delta}{=} \pi(-\theta) + r + \beta(1-\alpha) - \bar{\delta}$.

A straightforward calculation based on Lemma 4.20 (iii) shows that $\bar{\zeta}^* = \bar{\zeta}+1$. This allows us to conclude that

$$\frac{\mathbb{E}\exp\left(\frac{\alpha\bar{S}(\tau)}{1-\alpha}\right)}{\left\{\mathbb{E}^*\exp\left(\frac{\bar{S}(\tau^*)}{1-\alpha}\right)\right\}^\alpha} = \frac{\bar{\zeta}}{(\bar{\zeta}^*)^\alpha}\frac{(\frac{1}{1-\alpha}-\bar{\zeta}^*)^\alpha}{\frac{\alpha}{1-\alpha}-\bar{\zeta}} = \frac{\bar{\zeta}}{(\bar{\zeta}^*)^\alpha}\left(\frac{\alpha}{1-\alpha}-\bar{\zeta}\right)^{-(1-\alpha)}. \tag{4.49}$$

Using the definition of $\hat{\delta}$, one can show that $\hat{\zeta} \overset{\Delta}{=} \frac{\alpha}{1-\alpha}$ is the unique solution to

$$\pi(\theta\hat{\zeta}) + \hat{\mu}\hat{\zeta} = \hat{\delta} + \alpha\beta \tag{4.50}$$

with $\hat{\mu} \overset{\Delta}{=} \pi(-\theta) + r + \beta(1-\alpha) - \hat{\delta}$. Since, by definition, $\bar{\zeta}$ depends continuously on $\bar{\delta}$ and because equation (4.50) is the limit of the first equation in (4.48) for $\bar{\delta} \downarrow \hat{\delta}$, this shows that $\bar{\zeta} \to \hat{\zeta} = \frac{\alpha}{1-\alpha}$ as $\bar{\delta} \downarrow \hat{\delta}$. Now, the claimed convergence (4.47) can be read off equation (4.49). □

4.2.5 Case Studies

This section illustrates the preceding results by two case studies where X is either a Brownian motion or a Poisson process. For the special case of Brownian motion, our results allow to recover and extend the results by Hindy and Huang (1993). In particular, we will recover the singularity of optimal consumption plans in this case. By contrast, in the Poisson case, optimal consumption may occur in gulps and at rates. All results are stated under the standing Assumptions 4.4–4.7.

Geometric Brownian Motion

Let $X = (W(t),\ t \geq 0)$ be a Brownian motion. In this case, our optimization problem (4.20) is well–posed if and only if

$$\delta > \hat{\delta} = \alpha r + \frac{1}{2}\frac{\alpha\theta^2}{1-\alpha}\,. \tag{4.51}$$

Note that this is exactly the regularity assumption needed in the context of the classical Merton portfolio problem; compare, e.g., Karatzas and Shreve (1998), Remark 3.9.23, Merton (1990), Section 4.6, or Korn (1997), Corollary 3.3.7.

Recall that the result in Hindy and Huang (1993) is obtained by use of the Bellman methodology under the additional parameter restriction

$$\delta < r + \beta(1-\alpha)\,; \tag{4.52}$$

confer their equation (41). Our approach shows that this assumption can be dispensed with. We only need the natural condition (4.51); compare Theorem 4.4 and Proposition 4.23.

Let us now focus on the economic interpretation of the results in the Brownian case. Recall that the agent consumes whenever the process

$$A^K(t) = \eta \vee K^{-\frac{1}{1-\alpha}} \exp\left(\frac{1}{1-\alpha} \sup_{0\leq v\leq t}\{\theta X(v) + \mu v\}\right)$$

increases. Since X is Brownian motion, the drift μ of (4.30) is given by

$$\mu = \tfrac{1}{2}\theta^2 + r + \beta(1-\alpha) - \delta\,.$$

From the structure of A^K, we can immediately infer the following fundamental difference between the classic time–additive models and the Hindy–Huang-Kreps approach: there is no open time interval during which the Hindy–Huang–Kreps agent consumes all the time. Consumption occurs in a singular way, similar to the behavior of Brownian local time. This has already been pointed out by Hindy and Huang (1993). In their case, i.e., when (4.52) holds true, the process A^L diverges to infinity. Hence, the agent never refrains from consumption completely. In fact, our analysis shows that this is the case iff $\mu \geq 0$, i.e., iff

$$\delta \leq \tfrac{1}{2}\theta^2 + r + \beta(1-\alpha)\,.$$

It is interesting to see what happens if this inequality does not hold true. In this case, the overall supremum of the Brownian motion with drift $(\theta W(t) + \mu t,\ t \geq 0)$ is finite almost surely. Thus, there is an almost surely finite last time of consumption. However, since this is not a stopping time the agent will not consume all his wealth at that time because he does not know for sure that there will not be another opportunity for consumption! To illustrate this point further, let us calculate the optimal portfolio for an agent in a standard Samuelson–type model of the asset market.

Portfolios Consider a complete financial market with one risky asset whose price evolves according to

$$P(0) > 0, \quad dP(t) = P(t)\,(\sigma\, dW(t) + (r + \theta\sigma)\, dt) \quad (t \geq 0)$$

for some $\sigma > 0$. The agent uses the asset and the bond to finance his consumption plan C^K. Under $\mathbb{P}^*$,

$$W^*(t) \triangleq W(t) + \theta t \quad (t \geq 0)$$

becomes a Brownian motion and the discounted asset price $\bar{P} = (e^{-rt}P(t),\ t \geq 0)$ is — as usual — a $\mathbb{P}^*$–martingale with

$$d\bar{P}(t) = \sigma \bar{P}(t)\, dW^*(t) \quad (t \geq 0)\,.$$

Denote by

$$V^K(t) \triangleq \mathbb{E}^* \left[\int_t^{+\infty} e^{-r(s-t)}\, dC^K(s) \,\middle|\, \mathcal{F}_t \right]$$

the present value of the remaining consumption at time $t \geq 0$. The portfolio strategy π^K we are looking for has to satisfy

$$dV^K(t) = \pi^K(t)\, d\bar{P}(t) - e^{-rt}\, dC^K(t) \quad (t \geq 0)\,.$$

Proposition 4.24 *The agent puts a constant fraction of his wealth in the risky asset:*

$$\frac{\pi^K(t)\bar{P}(t)}{V^K(t)} \equiv \frac{\zeta^*\theta}{\sigma},$$

where ζ^* is as in (4.44).

Remark 4.25 *This similarity to the original Merton portfolio problem has already been observed by Hindy and Huang (1993) who proved this result using different methods based on their dynamic programming approach.*

Proof : We are interested in the representation of the martingale part of V^K as a stochastic integral with respect to W^*. To determine this representation, let us compute V^K explicitly.

We have $V^K(0-) = \Psi^K$, which has been calculated in (4.45). For $t > 0$ we proceed along the same lines as in the proof of Theorem 4.4 and in the calculation leading to (4.45) to obtain:

$$V^L(t) = \frac{e^{-\beta t}}{\beta}\left(\mathbb{E}^*\left[\int_t^{+\infty}(r+\beta)e^{-(r+\beta)(s-t)}A^K(s)\,ds\,\middle|\,\mathcal{F}_t\right] - A^K(t)\right).$$

The above expectation can be rewritten as

$$\mathbb{E}^*\left[A^L(t)\vee K^{-\frac{1}{1-\alpha}}\exp\frac{\theta W^*(t)+\mu^*t+\sup\limits_{0\le v\le\tau^*}\{\theta W^*(t+v)-\theta W^*(v)+\mu^* v\}}{1-\alpha}\,\middle|\,\mathcal{F}_t\right]$$

where τ^* is an independent exponential random variable with parameter $r+\beta>0$ and

$$\mu^* \triangleq -\frac{1}{2}\theta^2 + r + \beta(1-\alpha) - \delta\,.$$

The Markov property of Brownian motion and Lemma 4.20 (ii) allow us to conclude that the above conditional expectation is equal to

$$A^K(t) + \frac{K^{-\zeta^*}}{\nu}e^{\zeta^*\mu^* t}A^K(t)^{-\nu}e^{\zeta^*\theta W^*(t)}$$

where ζ^* is determined by (4.44) and $\nu \triangleq (1-\alpha)\zeta^*-1$, a strictly positive constant because of condition (4.51). The present value of the consumption policy C^K is therefore given by

$$V^K(t) = \frac{K^{-\zeta^*}}{\beta\nu}e^{(\zeta^*\mu^*-\beta)t}A^K(t)^{-\nu}e^{\zeta^*\theta W^*(t)}. \tag{4.53}$$

Hence,

$$dV^K(t) = V^K(t)\zeta^*\theta\,dW^*(t) + \text{terms of bounded variation}$$

and we deduce that at each time $t \geq 0$ the investor must hold

$$\pi^K(t) \triangleq \frac{\zeta^*\theta}{\sigma}\frac{V^K(t)}{\bar{P}(t)}$$

shares of the risky asset in his portfolio in order to finance the consumption policy C^K.

□

Remark 4.26 *If $\sigma = \theta\zeta^*$, the agent invests all his wealth in the risky asset. This case can be viewed as a single–agent equilibrium of the stock market for this type of investors.*

Consider again the case when there is an almost surely finite, yet imperceptible last time of consumption. This occurs, as we pointed out above, iff

$$\delta > \tfrac{1}{2}\theta^2 + r + \beta(1-\alpha)\,.$$

In this case, the investor's level of satisfaction eventually decreases at rate β, inducing an ever increasing appetite. His wealth, however, decreases at the higher rate $|\zeta^*\mu^*| + \beta$, as can be read off (4.53). Thus, the investor's relative level of satisfaction — the fraction of his level of satisfaction and his wealth — remains large. This in turn drives him to wait for better times to come. He keeps being engaged in the risky asset although he knows that with positive probability he may never consume again. This illustrates that, as already noted by Hindy and Huang (1993), an Hindy–Huang–Kreps investor is less risk averse than his time–additive counterpart, because he obtains utility from past consumption. Due to this effect, he can afford to invest in the risky asset and to refrain from consumption for a while in order to speculate on a higher future level of satisfaction.

Geometric Poisson Process

Let us now study Poisson price processes and put $X = (\pm N(t),\ t \geq 0)$. A jump of the process N corresponds to an unpredictable 'price shock' or, in the terminology of Hindy and Huang (1993), an 'information surprise'. We distinguish the two cases where the shocks are upward (price increase) or downward (price decrease).

Upward Price Shocks First we consider the case of upward price shocks, i.e., $X = (-N(t),\ t \geq 0)$, a Poisson process with downward jumps and intensity $\lambda > 0$ under the objective probability $\mathbb{P}$.

For this choice of X, the optimization problem (4.20) is well–posed iff

$$\delta > \hat{\delta} = \alpha r + \lambda\left((1-\alpha)e^{-\frac{\alpha\theta}{1-\alpha}} + \alpha e^{\theta} - 1\right)\,.$$

As before,

$$A^K(t) = \eta \vee K^{-\frac{1}{1-\alpha}} \exp\left(\frac{1}{1-\alpha} \sup_{0 \le v \le t} \{\theta X(v) + \mu v\} \right) ,$$

where now

$$\mu \stackrel{\Delta}{=} \lambda(e^{\theta} - 1) + r + \beta(1-\alpha) - \delta .$$

In contrast to the Brownian case, it now may happen that $X_{\theta,\mu} = (\theta X(t) + \mu t,\ t \ge 0)$ is a decreasing process. Indeed, this is the case iff $\mu \le 0$, i.e., iff

$$\delta \ge \lambda(e^{\theta} - 1) + r + \beta(1-\alpha) .$$

Hence, a very impatient agent (characterized by a high rate of time preference δ), optimally consumes his whole wealth by one single gulp at time $t = 0$. If the agent is not that impatient, then, apart from a possible initial gulp, he only consumes at rates

$$dC^K(t) = \frac{1}{\beta} e^{-\beta t} dA^K(t) = \frac{\lambda(e^{\theta} - 1) + r + \beta(1-\alpha) - \delta}{\beta(1-\alpha)} e^{-\beta t} A^K(t) 1_{\{\dot{A}^K(t) \neq 0\}}\, dt \quad (t > 0)$$

until an upward price shock makes him refrain from consumption. After a while, when his wealth and appetite have become large enough again, he restarts consumption until the next shock, etc.

Downward Price Shocks In the second Poisson example, there are downward price shocks, i.e., $X = (N(t),\ t \ge 0)$ with N as before.

As before,

$$A^K(t) = \eta \vee K^{-\frac{1}{1-\alpha}} \exp\left(\frac{1}{1-\alpha} \sup_{0 \le v \le t} \{\theta X(v) + \mu v\} \right)$$

where, in this case,

$$\mu \stackrel{\Delta}{=} \lambda(e^{-\theta} - 1) + r + \beta(1-\alpha) - \delta .$$

Observe that now X has positive jumps and, therefore, neither Assumption 4.8 (i) nor Assumption 4.8 (ii) holds true. Hence, the closed–form expressions for the prices of optimal consumption plans and their utilities as derived at the end of Section 4.2.4 are no longer valid here.

However, we still have that the utility maximization problem (4.20) is well–posed if

$$\delta > \hat{\delta} = \alpha r + \lambda \left((1-\alpha) e^{\frac{\alpha\theta}{1-\alpha}} + \alpha e^{-\theta} - 1 \right) . \tag{4.54}$$

Remark 4.27 *We conjecture, but cannot yet prove that condition* (4.54) *is also necessary for problem* (4.20) *to be well–posed in the case considered here. We know by Theorem 4.4 that the problem is ill–posed if $\delta < \hat{\delta}$. Thus the only open case is $\delta = \hat{\delta}$.*

Depending on the parameter values, two types of (optimal) consumption behavior can emerge in the presence of downward price shocks:

- If we have $\mu \geq 0$, then, once the investor has started consumption, he consumes continually at rates

$$\dot{C}^K(t) = \frac{\lambda(e^{-\theta} - 1) + r + \beta(1-\alpha) - \delta}{\beta(1-\alpha)} e^{-\beta t} A^K(t)$$

  and takes a gulp

$$\Delta C^K(t) = \frac{e^{\frac{\theta}{1-\alpha}}}{\beta} e^{-\beta t} A^K(t-) \Delta N(t)$$

  whenever a price shock occurs. This is due to the fact that prices decline very fast and the relative wealth of the consumer increases.

- If the world is not such a comfortable one, i.e., if $\mu < 0$, then the agent consumes only in gulps, namely every time a 'favorable' price shock causes A^K to reach a new maximum.

Chapter 5

General Equilibrium Theory

In this chapter, we prove existence of a general equilibrium in a multi–agent pure exchange economy when agents' preferences exhibit local substitution in the sense of Hindy, Huang, and Kreps (1992).

As usual in the context of infinite dimensional commodity spaces, the Negishi–method is the basis for the proof of the Existence Theorem 5.1. However, in contrast to the usual approach as, e.g., in Mas-Colell and Richard (1991), we do not restrict price functionals a priori to be continuous on the consumption space. This continuity is established only a posteriori in Theorem 5.2 under the additional assumptions that the information flow in the economy is quasi–leftcontinuous and that utility gradients are semimartingales with a continuous compensator. The general structure of supporting prices reveals that, without this extra structural requirements, there may actually be no continuous equilibrium price functionals for our economy.

If utility gradients are semimartingales, so are equilibrium prices since they take the form of a weighted maximum of such gradients. This is remarkable since in the standard time–additive model semimartingale prices can be ensured only by assuming that the aggregate endowment rate follows a semimartingale. In our context, price processes are semimartingales for *any* kind of endowment stream.

In contrast to a conjecture in Hindy and Huang (1992), however, it is not possible to ensure in general existence of an interest rate, i.e., absolute continuity of the compensator of the price semimartingale. Technically, this follows from an application of the Itô–Tanaka formula to the representation of equilibrium prices as the weighted maximum of utility gradients. It turns out that the predictable part of equilibrium prices typically has a singular local time component, even when the basic utility gradients are compensated by absolutely continuous processes. The economic reason for this singularity is that agents refrain from consuming when prices are too high as compared to their marginal utility of consumption, and only resume consumption after a decline of prices. Hence, it may happen that the identity of the agent whose gradient sets the price is changing when one agent stops consuming and another one restarts. If prices fluctuate in a diffusion–

like manner, such changes may occur arbitrarily fast, leading to a singular component in the evolution of equilibrium prices. A similar singularity effect can be observed in time–additive settings when marginal utility at zero is finite as has been shown by Karatzas, Lehoczky, and Shreve (1991).

Our approach to prove existence of equilibria relies on a Kuhn–Tucker characterization of efficient allocations and on the structure of supporting prices as weighted maxima of utility gradients. A central technical tool is Komlós' (1967) theorem. In its version by Kabanov (1999), this result gives us a powerful compactness principle which we use to prove both existence of efficient allocations and their continuous dependence on agents' weights. In conjunction with an argument going back to Bewley (1969), this continuity allows us to prove upper–hemicontinuity of the usual excess utility correspondence. Existence of equilibrium is then obtained by applying Kakutani's fixed point theorem.

5.1 The Economy

We consider a stochastic pure exchange economy with a finite set of agents $\mathcal{I}$. All agents share the same consumption space

$$\mathcal{X}_+ \triangleq \{C \in \mathcal{C} \mid \mathbb{E}C(\hat{T}) < +\infty\}\,.$$

On this space we define the metric

$$d_{\mathcal{X}_+}(C, C') \triangleq d_{\mathcal{C}}(C, C') + \mathbb{E}|C(\hat{T}) - C'(\hat{T})| \quad (C, C' \in \mathcal{X}_+)\,.$$

This metric endows $\mathcal{X}_+$ with the topology of weak convergence in probability plus $L^1(\mathbb{P})$–convergence of total masses. This slight topological strengthening of our general framework set up in Chapter 1 ensures enough integrability of our problem such that we no longer have to take care of possibly infinite utilities.

5.1.1 Endowments and Preferences

Each agent $i \in \mathcal{I}$ is endowed with some cumulative income stream $E_i \in \mathcal{X}_+$. To avoid trivial cases, we assume $E_i \neq 0$. Aggregate endowment is $E \triangleq \sum_i E_i$.

Assumption 5.1 *Agent i's preferences are described by a utility functional $V_i : \mathcal{X}_+ \to \mathbb{R}$ with the following properties:*

(i) V_i is continuous with respect to $d_{\mathcal{X}_+}$, strictly concave and strictly increasing with respect to $\preceq$.

(ii) *For every* $C \in \mathcal{X}_+$ *there exists a bounded optional process* $\nabla V_i(C)$ *with the subgradient property*

$$V_i(C') - V_i(C) \leq \langle \nabla V_i(C), C' - C \rangle \quad (C' \in \mathcal{X}_+)\,.$$

Subgradients are continuous in the sense that, for any two consumption plans $C, C' \in \mathcal{X}_+$*, we have*

$$\lim_{\varepsilon \downarrow 0} \langle \nabla V_i(\varepsilon C' + (1-\varepsilon)C), C' - C \rangle = \langle \nabla V_i(C), C' - C \rangle \ .$$

Hence, we assume essentially convexity of preferences and a sufficient degree of smoothness. In addition to Assumption 5.1, we require the following technical

Assumption 5.2 *Subgradients are uniformly bounded from above and bounded away from zero in the sense that there are nonnegative, optional processes* $b, B \in L^1_+(\mathbb{P} \otimes dE)\backslash\{0\}$ *such that*

$$b \leq \nabla V_i(C) \leq B \quad \mathbb{P} \otimes dE\text{–}a.e.$$

for all $C \in \mathcal{X}_+$ *with* $C \preceq E$.

Sufficient conditions for Assumptions 5.1 and 5.2 are given by

Lemma 5.1 *Suppose* V_i *takes the expected utility form* $V_i(.) = \mathbb{E}U_i(.)$ *where* U_i *satisfies Assumption 1.1. Assume moreover that the utility subgradient* ∇U_i *is pointwise decreasing with respect to* $\preceq$ *and that it is bounded from above:*

$$K \stackrel{\Delta}{=} \mathbb{P}\text{–}\operatorname{ess\,sup} \sup_{0 \leq t \leq \hat{T}} \nabla U_i(0)(t) < +\infty\,.$$

Then Assumptions 5.1 and 5.2 are satisfied.

Proof : Concavity and monotonicity of $V_i(.) = \mathbb{E}U_i(.)$ have been established in Proposition 1.4. To prove continuity, let $C^n \in \mathcal{X}_+$ $(n = 1, 2, \ldots)$ tend to $C^0 \in \mathcal{X}_+$ with respect to the metric $d_{\mathcal{X}_+}$. Then we have, in particular, $d_{\mathcal{C}}(C^n, C^0) \to 0$. Due to Proposition 1.4 (i), it therefore suffices to show uniform integrability of $(U_i(C^n),\ n = 1, 2, \ldots)$. This follows from the assumed $L^1(\mathbb{P})$–convergence $C^n(\hat{T}) \to C^0(\hat{T})$. In fact, Assumption 1.1 (ii) allows us to estimate

$$L^1(\mathbb{P}) \ni U_i(0) \leq U_i(C^n) \leq U_i(0) + (\nabla U_i(0), C^n) \leq U_i(0) + KC^n(\hat{T})\,.$$

Convergence of $(C^n(\hat{T}),\ n = 1, 2, \ldots)$ in $L^1(\mathbb{P})$ yields uniform $\mathbb{P}$–integrability of this sequence. By the above estimate this property indeed carries over to $(U_i(C^n),\ n = 1, 2, \ldots)$ as we wanted to show.

Existence of subgradients has also been established in Proposition 1.4. Monotonicity of subgradients entails

$$b \triangleq \nabla V(E) \leq \nabla V(C) \leq \nabla V(0) \triangleq B$$

for every $C \in \mathcal{X}_+$ with $C \preceq E$. For continuity of subgradients in the sense of Assumption 5.1 (ii), consider $C, C' \in \mathcal{X}_+$ and put $C^\varepsilon \triangleq \varepsilon C' + (1-\varepsilon)C$ for $\varepsilon \in (0,1)$. By Assumption 1.1 (iii)

$$(\nabla U(C^\varepsilon), C' - C) \to (\nabla U(C), C' - C) \quad (\varepsilon \downarrow 0)$$

almost surely. Since, in addition,

$$|(\nabla U(C^\varepsilon), C' - C)| \leq (\nabla U(0), C' + C) \in L^1(\mathbb{P})$$

we may use dominated convergence to deduce our assertion

$$\langle \nabla V(C^\varepsilon), C' - C \rangle \to \langle \nabla V(C), C' - C \rangle \quad (\varepsilon \downarrow 0)\,.$$

□

Remark 5.2 *For Hindy–Huang–Kreps preferences as introduced in Section 1.2.2,* strict *monotonicity and* strict *concavity hold true only on the slightly smaller consumption space $\{C \in \mathcal{X}_+ \mid \Delta C(\hat{T}) = 0\}$, since consumption made at time $t = \hat{T}$ obviously does not contribute to utility. However, strict concavity and strict monotonicity will only be needed for plans $C \in \mathcal{X}_+$ satisfying $C \preceq E$. Hence, this minor deviation from Assumption 5.1 does not impose any problems if we assume $\Delta E(\hat{T}) = 0$.*

Every bounded measurable process $\psi : \Omega \times [0, \hat{T}] \to \mathbb{R}$ gives rise to a (not necessarily continuous) linear functional $\langle \psi, . \rangle$ on $\mathcal{X}_+$. If, in addition, ψ is nonnegative and optional, we call it a price process, and we call $\langle \psi, . \rangle$ a price functional on $\mathcal{X}_+$.

Remark 5.3 *Note that we do not assume continuity of price functionals a priori as is usually done in equilibrium theory. In fact, we will first identify an equilibrium price functional in the much larger space of all linear functionals and establish only a posteriori its continuity under appropriate conditions; see Theorem 5.2.*

5.1.2 Equilibrium

An *allocation* is a vector $(C_i)_{i \in \mathcal{I}} \in \mathcal{X}_+^{\mathcal{I}}$. It is *feasible* if $\sum_i C_i \preceq E$. The set of feasible allocations will be denoted by $\mathcal{Z}$.

An (Arrow–Debreu) *equilibrium* consists of a feasible allocation $(C_i^*)_{i \in \mathcal{I}} \in \mathcal{Z}$ and a price process ψ^* such that, for any $i \in \mathcal{I}$, the consumption plan C_i^* maximizes agent i's utility over all $C_i \in \mathcal{X}_+$ satisfying the budget–constraint $\langle \psi^*, C_i \rangle \leq \langle \psi^*, E_i \rangle$.

The main theorem of this chapter is

Theorem 5.1 *Under Assumptions 5.1 and 5.2, an equilibrium exists.*

Our strategy of proof is based on the Negishi method as usual with infinite dimensional commodity spaces. In Section 5.2.1, we first characterize the unique efficient allocation for any given vector of agents' weights. Existence of efficient allocations is ensured by Komlós' theorem. The same theorem is also useful in showing that the efficient allocation depends continuously on agents' weights.

In Section 5.2.2, we describe the price functionals which support the efficient allocations on the order ideal. The corresponding price processes are given as maxima of agents' utility gradients at the efficient allocation. This particular structure allows us to extend the supporting price functionals from the order ideal to the whole consumption space.

Finally, we define in Section 5.3 the usual excess utility correspondence. Upper hemicontinuity of this correspondence follows from a classical argument due to Bewley (1969), and Kakutani's fixed point theorem yields existence of an equilibrium.

5.2 Efficient Allocations and Supporting Prices

The first step in our program is to characterize efficient allocations. To this end, we prove a version of the Kuhn–Tucker Theorem for the welfare maximization problem. As usual, if agents consume, their utility gradients are equalized in an efficient allocation. The common value for the utility gradients is the Lagrange multiplier for this problem. At the same time, it gives rise to a price functional which supports the efficient allocation in the sense of the Second Welfare Theorem. Of course, their may be other functionals with the same supporting property. However, as we shall see, they all share the same structure.

5.2.1 The Social Welfare–Problem

Let us introduce the set of normalized weights

$$\Lambda \triangleq \left\{ \lambda \in \mathbb{R}_+^{\mathcal{I}} \;\middle|\; \sum\nolimits_i \lambda_i = 1 \right\} .$$

An allocation $(C_i)_{i\in\mathcal{I}}$ is called λ–efficient for agents' weights $\lambda \in \Lambda$ if it maximizes the social welfare $\sum_i \lambda_i V_i(C_i)$ subject to the feasibility constraint

$$\sum\nolimits_i C_i \preceq E .$$

The characterization of efficient allocations is achieved by the following Kuhn–Tucker–like result.

Lemma 5.4 *For any $\lambda \in \Lambda$, there exists a unique λ–efficient allocation $(C_i^\lambda)_{i\in\mathcal{I}} \in \mathcal{Z}$. It is characterized by the joint validity of the following properties (i)–(iii) for some nonnegative, optional random variable ψ:*

(i) $\sum_i C_i^\lambda = E$,

(ii) $\lambda_i \nabla V_i(C_i^\lambda) \leq \psi$,

(iii) $\langle \psi, C_i^\lambda \rangle = \langle \lambda_i \nabla V_i(C_i^\lambda), C_i^\lambda \rangle$ *for every* $i \in \mathcal{I}$.

The random variable ψ plays the role of a Lagrange multiplier for the problem of maximizing social welfare. By the flat–off condition (iii), it is uniquely determined $\mathbb{P} \otimes dE$–almost everywhere as

$$\psi = \max_i \{\lambda_i \nabla V_i(C_i^\lambda)\}\,.$$

PROOF : Uniqueness of the λ–efficient allocation follows as usual from the strict concavity of the utility functionals V_i $(i \in \mathcal{I})$ by considering a convex combination of two λ–efficient allocations.

To prove existence, we choose a sequence of feasible allocations $((C_i^n)_{i\in\mathcal{I}},\ n = 1, 2, \ldots)$ which asymptotically maximizes social welfare in the sense that

$$\lim_n \sum\nolimits_i \lambda_i V_i(C_i^n) = \sup_{(C_i)_{i\in\mathcal{I}}\in\mathcal{Z}} \sum\nolimits_i \lambda_i V_i(C_i)\,.$$

By feasibility, each sequence $(C_i^n(\hat{T}),\ n = 1, 2, \ldots)$ $(i \in \mathcal{I})$ is bounded in $L^1(\mathbb{P})$. Hence, we may use Kabanov's version of Komlós' theorem (Kabanov (1999), Lemma 3.5; Komlós (1967)) to obtain existence of a subsequence, again denoted by n, such that each sequence $(C_i^n,\ n = 1, 2, \ldots)$ $(i \in \mathcal{I})$ is almost surely weakly Cesaro convergent to some $C_i^* \in \mathcal{X}_+$ $(i \in \mathcal{I})$, i.e., we have almost surely

$$\tilde{C}_i^n(t) \triangleq \frac{1}{n} \sum\nolimits_{k=1}^n C_i^k(t) \to C_i^*(t)$$

for $t = \hat{T}$ and also for every point of continuity t of C_i^*. The above convergence entails in particular that also $(C_i^*)_{i\in\mathcal{I}}$ is a feasible allocation. Moreover, it implies $d_{\mathcal{X}_+}(\tilde{C}_i^n, C_i^*) \to 0$, as, in addition to the above weak convergence, the sequence $(\tilde{C}_i^n(\hat{T}),\ n = 1, 2, \ldots)$ is dominated by $E(\hat{T}) \in L^1(\mathbb{P})$. Finally, also $((\tilde{C}_i^n)_{i\in\mathcal{I}},\ n = 1, 2, \ldots)$ is a maximizing sequence of feasible allocations due to concavity of social welfare. As $\tilde{C}_i^n \to C_i^*$ in $(\mathcal{X}_+, d_{\mathcal{X}_+})$ for every $i \in \mathcal{I}$, this implies λ–efficiency of $(C_i^*)_{i\in\mathcal{I}}$ by continuity of preferences (Assumption 5.1 (i)).

In order to prove the asserted characterization of efficient allocations, we proceed in three steps.

1. We start with sufficiency of (i)–(iii). Let $(C_i^*)_{i\in\mathcal{I}}$ be an allocation satisfying these conditions and let $(C_i)_{i\in\mathcal{I}} \in \mathcal{Z}$ be another feasible allocation. Due to the subgradient estimate of Assumption 5.1 (ii), we have

$$\sum_i \lambda_i\{V_i(C_i^*) - V_i(C_i)\} \geq \sum_i \lambda_i \langle \nabla V_i(C_i^*), C_i^* - C_i\rangle$$

which by (ii) and (iii) is

$$\geq \left\langle \psi, \sum_i C_i^* - \sum_i C_i \right\rangle .$$

This yields

$$\sum_i \lambda_i\{V_i(C_i^*) - V_i(C_i)\} \geq \left\langle \psi, E - \sum_i C_i \right\rangle \geq 0$$

by nonnegativity of ψ in conjunction with condition (i) and feasibility of $(C_i)_{i\in\mathcal{I}}$. Hence, an allocation $(C_i^*)_{i\in\mathcal{I}}$ with (i)–(iii) indeed attains maximal social welfare among all feasible allocations, given agents' weights λ.

2. Necessity of condition (i) follows immediately from strict monotonicity of preferences. To prove that conditions (ii) and (iii) hold true for some Lagrange multiplier ψ, consider another feasible allocation $(C_i)_{i\in\mathcal{I}} \in \mathcal{Z}$. For $\varepsilon \in [0,1]$, let $C_i^\varepsilon \triangleq \varepsilon C_i + (1-\varepsilon) C_i^\lambda$ $(i \in \mathcal{I})$. Since every allocation $(C_i^\varepsilon)_{i\in\mathcal{I}}$ is feasible, λ–efficiency of $(C_i^\lambda)_{i\in\mathcal{I}}$ yields, for $\varepsilon \in (0,1]$,

$$\begin{aligned} 0 &\geq \frac{1}{\varepsilon}\sum_i \lambda_i\{V_i(C_i^\varepsilon) - V_i(C_i^\lambda)\} \\ &\geq \frac{1}{\varepsilon}\sum_i \lambda_i \left\langle \nabla V_i(C_i^\varepsilon), C_i^\varepsilon - C_i^\lambda \right\rangle = \sum_i \lambda_i \left\langle \nabla V_i(C_i^\varepsilon), C_i - C_i^\lambda \right\rangle . \end{aligned}$$

Using continuity of subgradients (Assumption 5.1 (ii)), we may let $\varepsilon \downarrow 0$ in the above estimate to deduce

$$\sum_i \langle \phi_i, C_i\rangle \leq \sum_i \left\langle \phi_i, C_i^\lambda \right\rangle$$

where $\phi_i \triangleq \lambda_i \nabla V_i(C_i^\lambda)$ $(i \in \mathcal{I})$.

We see that $(C_i^\lambda)_{i\in\mathcal{I}}$ also solves the linear problem to maximize $\sum_i \langle \phi_i, C_i\rangle$ over all feasible allocations. In Step 3 below, we show that every solution $(C_i^*)_{i\in\mathcal{I}}$ of this problem satisfies $\langle \psi, C_i^*\rangle = \langle \phi_i, C_i^*\rangle$ for every $i \in \mathcal{I}$ where $\psi \triangleq \max_i \phi_i$. For $C^* = C^\lambda$, we find that, with this choice of ψ, conditions (ii) and (iii) hold true as well.

3. Let $(C_i^*)_{i\in\mathcal{I}} \in \mathcal{Z}$ be a feasible allocation such that

$$\sum\nolimits_i \langle \phi_i, C_i\rangle \leq \sum\nolimits_i \langle \phi_i, C_i^*\rangle$$

for every other feasible allocation $(C_i)_{i\in\mathcal{I}} \in \mathcal{Z}$.

Consider the allocation defined by the optional random measures

$$dC_i(t) \triangleq n(t)^{-1} 1_{\{\phi_i(t)=\psi(t)\}} dE(t) \quad (i \in \mathcal{I})$$

where $\psi(t) = \max_i \phi_i(t)$ and where $n(t) \triangleq \sum_i 1_{\{\phi_i(t)=\psi(t)\}}$ denotes the number of indices i realizing the maximum in $\max_i \phi_i(t)$ $(0 \leq t \leq \hat{T})$. Clearly, $(C_i)_{i\in\mathcal{I}}$ is feasible and satisfies

$$\sum\nolimits_i \langle \phi_i, C_i\rangle = \langle \psi, E\rangle \ .$$

Therefore, $\sum_i \langle \phi_i, C_i^*\rangle$ cannot be less than $\langle \psi, E\rangle$. On the other hand, since $\psi \geq \phi_i \geq 0$ for each $i \in \mathcal{I}$ and as $(C_i^*)_{i\in\mathcal{I}}$ is feasible, $\sum_i \langle \phi_i, C_i^*\rangle$ cannot be greater than $\langle \psi, E\rangle$. Thus, we must in fact have

$$\sum\nolimits_i \langle \phi_i, C_i^*\rangle = \langle \psi, E\rangle$$

which can hold true only if $\langle \psi, C_i^*\rangle = \langle \phi_i, C_i^*\rangle$ for every $i \in \mathcal{I}$.

□

The dependence of efficient allocations and induced expected utilities on agents' weights will be described in the following Lemma 5.5 and its Corollary 5.6. The latter result will be a corner stone for our proof of Theorem 5.1.

Lemma 5.5 *The mapping $\lambda \mapsto C^\lambda$ is weakly continuous in the sense that*

$$\lim_{\lambda^n\to\lambda^0} \left\langle \psi, C_i^{\lambda^n}\right\rangle = \left\langle \psi, C_i^{\lambda^0}\right\rangle \quad (i \in \mathcal{I})$$

for every $\psi \in L^1(\mathbb{P} \otimes dE)$.

PROOF : Let λ^n $(n = 1, 2, \ldots)$ tend to λ^0 in Λ. Put $C^n \triangleq C^{\lambda^n}$ and consider the densities

$$D_i^n \triangleq \frac{dC_i^n}{dE} \quad (n = 0, 1, \ldots,\ i \in \mathcal{I})\,.$$

Due to feasibility of efficient allocations, these densities have optional versions taking values in $[0, 1]$. Now, writing

$$\langle \psi, C_i^n\rangle = \mathbb{E}\int_0^{\hat{T}} \psi(t) D_i^n(t)\, dE(t) \quad (n = 0, 1, \ldots,\ i \in \mathcal{I})$$

for $\psi \in L^1(\mathbb{P} \otimes dE)$, we see that our assertion is equivalent to the assertion that, for every $i \in \mathcal{I}$ the densities D_i^n $(n = 1, 2, \ldots)$ converge to D_i^0 in the weak*–topology $\sigma\left(L^\infty(\mathbb{P} \otimes dE), L^1(\mathbb{P} \otimes dE)\right)$. As the unit ball in $L^\infty(\mathbb{P} \otimes dE)$ is sequentially compact with respect to this topology, this convergence will be proved once we know that, for each $i \in \mathcal{I}$, all weak*–convergent subsequences of $(D_i^n,\ n = 1, 2, \ldots)$ have the same limit D_i^0.

To prove this, we slightly abuse notation and suppose that $((D_i^n)_{i\in\mathcal{I}},\ n = 1, 2, \ldots)$ is a subsequence such that each component $(D_i^n,\ n = 1, 2, \ldots)$ $(i \in \mathcal{I})$ is weak*–convergent to some $D_i^\infty \in L^\infty(\mathbb{P} \otimes dE)$. We then have to show that $D_i^\infty = D_i^0$ for every $i \in \mathcal{I}$.

For this, we note first that, by feasibility, we have $\mathbb{E}C_i^n(\hat{T}) \leq \mathbb{E}E(\hat{T})$ for any $n = 1, 2, \ldots$ and every $i \in \mathcal{I}$. Hence, by Kabanov's version of Komlós' theorem, there is a subsequence, again denoted by $((C_i^n)_{i\in\mathcal{I}},\ n = 1, 2, \ldots)$, such that, for any $i \in \mathcal{I}$,

$$\tilde{C}_i^n \triangleq \frac{1}{n} \sum\nolimits_{k=1}^n C_i^k$$

almost surely converges weakly to some $C_i^* \in \mathcal{X}_+$ in the sense that

$$d_{\mathcal{M}_+}(\tilde{C}_i^n, C_i^*) \to 0 \quad (n \uparrow +\infty)\,.$$

By dominated convergence, we also have $\tilde{C}_i^n(\hat{T}) \to C_i^*(\hat{T})$ in $L^1(\mathbb{P})$ and, therefore, even $d_{\mathcal{X}_+}(\tilde{C}_i^n, C_i^*) \to 0$ for every $i \in \mathcal{I}$. Clearly, the densities

$$\tilde{D}_i^n \triangleq \frac{d\tilde{C}_i^n}{dE} = \frac{1}{n} \sum\nolimits_{k=1}^n D_i^k \quad (n = 1, 2, \ldots)$$

inherit weak*–convergence to D_i^∞ from $(D_i^n,\ n = 1, 2, \ldots)$. Therefore, each C_i^* is almost surely absolutely continuous with respect to E with density D_i^∞. Hence, in order to conclude our claim, we only need to prove that $(C_i^*)_{i\in\mathcal{I}}$ is the unique (!) efficient allocation for agents' weights λ^0.

To this end, consider any other feasible allocation $(C_i)_{i\in\mathcal{I}} \in \mathcal{Z}$. By continuity of preferences, we have

$$\sum\nolimits_i \lambda_i^0 V_i(C_i^*) = \lim_n \sum\nolimits_i \lambda_i^0 V_i(\tilde{C}_i^n)$$

which, due to the concavity of every $V_i(.)$ $(i \in \mathcal{I})$, is

$$\begin{aligned} &\geq \limsup_n \frac{1}{n} \sum\nolimits_{k=1}^n \sum\nolimits_i \lambda_i^0 V_i(C_i^k) \\ &= \limsup_n \frac{1}{n} \sum\nolimits_{k=1}^n \sum\nolimits_i \left\{\lambda_i^k V_i(C_i^k) + R_i^k\right\} \end{aligned}$$

for $R_i^k \triangleq (\lambda_i^0 - \lambda_i^k) V_i(C_i^k)$ $(k = 1, 2, \ldots, i \in \mathcal{I})$. This term tends to zero for $k \uparrow +\infty$ as $V_i(C_i^k) \in [V_i(0), V_i(E)]$ is uniformly bounded and $\lambda^k \to \lambda^0$ $(k \uparrow +\infty)$. Hence, we obtain

$$\sum\nolimits_i \lambda_i^0 V_i(C_i^*) \geq \limsup_n \frac{1}{n} \sum\nolimits_{k=1}^n \sum\nolimits_i \lambda_i^k V_i(C_i^k)$$

By λ^k–efficiency of allocation C^k $(k = 1, 2, \ldots)$, this is in turn

$$\begin{aligned} &\geq \limsup_n \frac{1}{n} \sum\nolimits_{k=1}^n \sum\nolimits_i \lambda_i^k V_i(C_i) \\ &= \limsup_n \sum\nolimits_i \left(\frac{1}{n} \sum\nolimits_{k=1}^n \lambda_i^k \right) V_i(C_i) \\ &= \sum\nolimits_i \lambda_i^0 V_i(C_i) \end{aligned}$$

where the last equality is due to the convergence $\lambda^n \to \lambda^0$. Hence, we have shown

$$\sum\nolimits_i \lambda_i^0 V_i(C_i^*) \geq \sum\nolimits_i \lambda_i^0 V_i(C_i)$$

for every feasible allocation $(C_i)_{i\in\mathcal{I}}$. Since, in addition, $(C_i^*)_{i\in\mathcal{I}}$ is feasible, it must coincide with the unique λ^0–efficient allocation $(C_i^0)_{i\in\mathcal{I}}$ and we are done. □

Let us note the following crucial

Corollary 5.6 *Every mapping $\lambda \mapsto V_i(C_i^\lambda)$ $(i \in \mathcal{I})$ is upper–semicontinuous, i.e.,*

$$\limsup_{\lambda^n \to \lambda^0} V_i(C_i^{\lambda^n}) \leq V_i(C_i^{\lambda^0}) .$$

PROOF : By concavity we have

$$V_i(C_i^{\lambda^n}) - V_i(C_i^{\lambda^0}) \leq \left\langle \nabla V_i(C_i^{\lambda^0}), C_i^{\lambda^n} - C_i^{\lambda^0} \right\rangle .$$

By Lemma 5.5, the last term tends to zero if $\lambda^n \to \lambda^0$ in Λ and we are done. □

5.2.2 Supporting Prices

For some of the following more technical arguments, it will be convenient to work with the auxiliary consumption space given by the order ideal

$$\tilde{\mathcal{X}}_+ = \tilde{\mathcal{X}}_+(E) \triangleq \left\{ X \in \mathcal{X}_+ \;\middle|\; \frac{dX}{dE} \text{ exists } \mathbb{P}\text{–a.s. and is } \mathbb{P} \otimes dE\text{–essentially bounded} \right\} .$$

For a more detailed discussion of this concept in general equilibrium theory, we refer the reader to Mas-Colell and Zame (1991).

Remark 5.7 *The space $\tilde{\mathcal{X}}_+$ can be identified with $L_+^\infty(\Omega \times [0, \hat{T}], \mathcal{O}, \mathbb{P} \otimes dE)$, the set of all nonnegative, optional processes which are $\mathbb{P} \otimes dE$–essentially bounded. Clearly, any feasible allocation is contained in $\tilde{\mathcal{X}}_+^{\mathcal{I}}$.*

By Lemma 5.4, we may associate with every efficient allocation $\left(C_i^\lambda\right)_{i\in\mathcal{I}}$ $(\lambda \in \Lambda)$ the Lagrange multiplier

$$\psi_\lambda \triangleq \max_i \{\lambda_i \nabla V_i(C_i^\lambda)\}\,.$$

We will have a lot more to say about the structure of these multipliers in Section 5.4. For the moment, let us content ourselves by noting that each ψ_λ can be viewed as a nonnegative, optional random variable in $L^1(\mathbb{P}\otimes dE)$. Hence, each of these Lagrange multipliers gives rise to a price density on $\tilde{\mathcal{X}}_+$. Considering ψ_λ as a price density is also sustained by the fact that it supports its associated efficient allocation $(C_i^\lambda)_{i\in\mathcal{I}}$. This will be proved in Lemma 5.9 below.

Beforehand, let us recall that a price density $\psi^* \in L^1(\mathbb{P}\otimes dE)$ supports an allocation $(C_i^*)_{i\in\mathcal{I}}$ with $\sum_i C_i^* = E$, if it is non–zero and if any preferred allocation $(C_i)_{i\in\mathcal{I}} \in \tilde{\mathcal{X}}_+^{\mathcal{I}}$ has a higher 'price' under ψ^* than $(C_i^*)_{i\in\mathcal{I}}$. More precisely, we say ψ^* supports $(C_i^*)_{i\in\mathcal{I}}$ with $\sum_i C_i^* = E$ iff $\langle\psi^*, E\rangle \neq 0$ and

$$V_i(C_i) \geq V_i(C_i^*),\ C_i \in \tilde{\mathcal{X}}_+ \text{ for all } i\in\mathcal{I} \text{ implies } \left\langle \psi^*, \sum\nolimits_i C_i \right\rangle \geq \langle \psi^*, E\rangle\ .$$

We note the following

Proposition 5.8 *If an allocation $(C_i^*)_{i\in\mathcal{I}}$ with $\sum_i C_i^* = E$ is supported by $\psi^* \in L^1(\mathbb{P}\otimes E)$, then $\psi^* > 0$ $\mathbb{P}\otimes dE$–a.e.*

Proof : Suppose to the contrary that $\mathbb{P}\otimes dE[\psi^* \leq 0] > 0$. Then, by strict monotonicity and continuity of preferences, we may choose $\varepsilon > 0$ small enough such that the allocation defined by

$$dC_i \triangleq (1-\varepsilon)\, dC_i^* + 1_{\{\psi^*\leq 0\}}\, dE \quad (i \in \mathcal{I})$$

is preferred to $(C_i^*)_{i\in\mathcal{I}}$. Thus, the supporting property of ψ^* implies

$$\langle\psi^*, E\rangle \leq \left\langle \psi^*, \sum\nolimits_i C_i\right\rangle = (1-\varepsilon)\,\langle\psi^*, E\rangle + |\mathcal{I}|\,\mathbb{E}\int_0^{\hat{T}} \psi^* 1_{\{\psi^*\leq 0\}}\, dE\,.$$

Hence, we obtain

$$|\mathcal{I}|\,\mathbb{E}\int_0^{\hat{T}} \psi^* 1_{\{\psi^*\leq 0\}}\, dE \geq \varepsilon\,\langle\psi^*, E\rangle > 0,$$

a contradiction to the obvious relation $\mathbb{E}\int_0^{\hat{T}} \psi^* 1_{\{\psi^*\leq 0\}}\, dE \leq 0$. □

Now, we may prove

Lemma 5.9 *The Lagrange multiplier ψ_λ supports its associated λ–efficient allocation $(C_i^\lambda)_{i\in\mathcal{I}}$. Moreover, any other optional price density $\psi \in L^1(\mathbb{P}\otimes dE)$ with this property is of the form*

$$\psi = \max_i\{k_i \nabla V_i(C_i^\lambda)\} \quad \mathbb{P}\otimes dE\text{–a.s.}$$

for some constants $k_i \geq 0$ $(i \in \mathcal{I})$, and the λ–efficient allocation satisfies the flat–off conditions

$$\left\langle \psi, C_i^\lambda \right\rangle = \left\langle k_i \nabla V_i(C_i^\lambda), C_i^\lambda \right\rangle \quad (i \in \mathcal{I})\,.$$

PROOF : In order to show that ψ_λ supports $\left(C_i^\lambda\right)_{i\in\mathcal{I}}$, take an allocation $(C_i)_{i\in\mathcal{I}} \in \tilde{\mathcal{X}}_+^{\mathcal{I}}$ with $V_i(C_i) \geq V_i(C_i^\lambda)$. Concavity of V_i yields

$$0 \leq \sum\nolimits_i \lambda_i \left\{V_i(C_i) - V_i(C_i^\lambda)\right\} \leq \sum\nolimits_i \lambda_i \left\langle \nabla V_i(C_i^\lambda), C_i - C_i^\lambda \right\rangle\,.$$

By properties (ii) and (iii) of the efficient allocation $(C_i^\lambda)_{i\in\mathcal{I}}$ (Lemma 5.4), the latter quantity is

$$\leq \sum\nolimits_i \left\langle \psi_\lambda, C_i - C_i^\lambda \right\rangle = \left\langle \psi_\lambda, \sum\nolimits_i C_i - E \right\rangle\,.$$

Hence, $\langle \psi_\lambda, \sum_i C_i\rangle \geq \langle \psi_\lambda, E\rangle$ which is the claimed supporting property.

For the second part of the lemma, suppose that $\psi \in L^1(\mathbb{P}\otimes dE)$ is optional and supports the allocation $(C_i^\lambda)_{i\in\mathcal{I}}$. We only need to show that there are nonnegative constants k_i $(i \in \mathcal{I})$ such that

$$k_i \nabla V_i(C_i^\lambda) \leq \psi \;\; \mathbb{P}\otimes dE\text{–a.e. and } \left\langle \psi, C_i^\lambda \right\rangle = \left\langle k_i \nabla V_i(C_i^\lambda), C_i^\lambda \right\rangle \tag{5.1}$$

for every $i \in \mathcal{I}$. To this end, put

$$k_i \triangleq \mathbb{P}\otimes dE\text{–}\operatorname{ess\,inf} \frac{\psi}{\nabla V_i(C_i^\lambda)} \quad (i \in \mathcal{I})\,.$$

Obviously, $k_i \geq 0$ for every $i \in \mathcal{I}$ and, of course, the first condition in (5.1) is satisfied. To verify the second condition for $i \in \mathcal{I}$, let us distinguish the cases $\left\langle \psi, C_i^\lambda \right\rangle = 0$ and $\left\langle \psi, C_i^\lambda \right\rangle > 0$.

In the first case, we may conclude from Proposition 5.8 that $C_i^\lambda = 0$. Thus, the second condition in (5.1) is satisfied trivially. For the case $\left\langle \psi, C_i^\lambda \right\rangle > 0$, we prove below that C_i^λ maximizes utility over all consumption plans $C_i \in \tilde{\mathcal{X}}_+$ with $\langle \psi, C_i\rangle \leq \left\langle \psi, C_i^\lambda \right\rangle$. Lemma 5.10 then yields the validity of the second condition in (5.1) also in this case.

To obtain the claimed optimality of C_i^λ, it suffices to show that any $C_i \in \tilde{\mathcal{X}}_+$ with $V_i(C_i) > V_i(C_i^\lambda)$ must satisfy $\langle \psi, C_i\rangle > \left\langle \psi, C_i^\lambda \right\rangle$. Note first that, for such a C_i, we also have $V_i((1-\varepsilon)C_i) > V_i(C_i^\lambda)$ for any sufficiently small $\varepsilon > 0$ by continuity V_i. Consider the allocation $(\tilde{C}_j)_{j\in\mathcal{I}}$ defined by

$$\tilde{C}_j \triangleq \begin{cases} (1-\varepsilon)C_i & \text{for } j = i, \\ C_j^\lambda & \text{otherwise.} \end{cases}$$

This allocation is preferred to $(C_i^\lambda)_{i\in\mathcal{I}}$ and, thus, the supporting property of ψ yields

$$\langle \psi, E\rangle \leq \left\langle \psi, \sum\nolimits_j \tilde{C}_j \right\rangle = \langle \psi, E\rangle + \left\langle \psi, (1-\varepsilon)C_i - C_i^\lambda \right\rangle .$$

This implies

$$0 < \left\langle \psi, C_i^\lambda \right\rangle \leq \langle \psi, (1-\varepsilon)C_i\rangle < \langle \psi, C_i\rangle$$

as we wanted to show. □

The following result was needed in the proof of Lemma 5.9:

Lemma 5.10 *Let $\psi \in L^1(\mathbb{P}\otimes dE)$ be an optional price density and assume V is a utility functional satisfying Assumption 5.1.*

If $C^* \in \tilde{\mathcal{X}}_+$ maximizes $V(C)$ over all $C \in \tilde{\mathcal{X}}_+$ with $\langle \psi, C\rangle \leq w$ then C^* meets the flat–off condition

$$\langle \psi, C^*\rangle = \langle k\nabla V(C^*), C^*\rangle$$

where

$$k \triangleq \mathbb{P}\otimes dE\text{–}\operatorname{ess\,inf} \frac{\psi}{\nabla V(C^*)} .$$

Proof : We will prove this lemma using arguments similar to the proof of the necessity part in Lemma 5.4.

1. Let C^* be an optimal consumption plan as above and let $C \in \tilde{\mathcal{X}}_+$ satisfy $\langle \psi, C\rangle \leq w$. Note that, for any $0 \leq \varepsilon \leq 1$, the plan $C^\varepsilon \triangleq \varepsilon C + (1-\varepsilon)C^*$ is also in $\tilde{\mathcal{X}}_+$ and satisfies $\langle \psi, C^\varepsilon\rangle \leq w$. By optimality of C^* and because of the subgradient property, we have

   $$0 \geq \frac{1}{\varepsilon}\{V(C^\varepsilon) - V(C^*)\} \geq \langle \nabla V(C^\varepsilon), C - C^*\rangle$$

   Due to Assumption 5.1 (ii), we may let $\varepsilon \downarrow 0$ in this estimate to infer

   $$\langle \nabla V(C^*), C\rangle \leq \langle \nabla V(C^*), C^*\rangle .$$

   Thus, putting $\phi^* \triangleq \nabla V(C^*) \geq 0$, we find that C^* also solves the linear problem to maximize $\langle \phi^*, C\rangle$ over all $C \in \tilde{\mathcal{X}}_+$ with $\langle \psi, C\rangle \leq w$, i.e., C^* is a solution to

   $$\max_{C\in\tilde{\mathcal{X}}_+,\ \langle\psi,C\rangle\leq w} \langle \phi^*, C\rangle .$$

2. We claim that the value of the preceding linear problem is $v = w/k$, where $k \triangleq \mathbb{P}\otimes dE-\operatorname{ess\,inf} \psi/\phi^*$.

   Indeed, since $k\phi^* \leq \psi$ $\mathbb{P}\otimes dE$–a.e. by definition of k,

   $$k\,\langle\phi^*, C\rangle \leq \langle\psi, C\rangle \leq w \quad \text{for every } C \text{ qualifying in the above max.}$$

   Thus, we have $kv \leq w$. To prove equality, consider for $K > k$ the consumption plan C^K defined by

   $$dC^K \triangleq c^K 1_{\{\psi/\phi^* \leq K\}}\, dE$$

   where the constant $c^K > 0$ is chosen such that

   $$\left\langle\psi, C^K\right\rangle = c^K \mathbb{E}\int_0^{\hat{T}} \psi 1_{\{\psi/\phi^* \leq K\}}\, dE = w\,.$$

   Note that this choice is indeed possible since $\mathbb{P}\otimes dE[\psi/\phi^* \leq K] > 0$ and $\psi > 0$ $\mathbb{P}\otimes dE$–a.e. by Proposition 5.8. Obviously, $C^K \in \tilde{\mathcal{X}}_+$ and

   $$\left\langle\phi^*, C^K\right\rangle = c^K \mathbb{E}\int_0^{\hat{T}} \phi^* 1_{\{\psi/\phi^* \leq K\}}\, dE \geq c^K \mathbb{E}\int_0^{\hat{T}} \psi 1_{\{\psi/\phi^* \leq K\}}\, dE/K = w/K\,.$$

   Now, let $K \downarrow k$ to derive the converse inequality $v \geq w/k$.

3. Combining Steps 1 and 2, we find

   $$\langle\nabla V(C^*), C^*\rangle = \max_{C\in\tilde{\mathcal{X}}_+,\, \langle\psi, C\rangle \leq w} \langle\phi^*, C\rangle = w/k = \langle\psi, C^*\rangle/k$$

   which is the desired flat–off condition.

□

5.3 Existence of Equilibria

After these technical preliminaries, we are now in a position to prove existence of equilibria for intertemporal consumption.

Proof of Theorem 5.1 We start by defining the correspondence $\mathcal{G}$ to which Kakutani's fixed point theorem will be applied. To this end, let, for any $\lambda \in \Lambda$, $\mathcal{S}(\lambda)$ denote the set of all optional price densities $\psi \in L^1(\mathbb{P}\otimes dE)$ which support the allocation C^λ and which, in addition, satisfy

$$\frac{b}{|\mathcal{I}|\,\langle B, E\rangle} \leq \psi \leq \frac{B}{\langle B, E\rangle} \quad \mathbb{P}\otimes dE\text{–a.e.} \quad \text{and} \quad \left\langle\psi, C_i^\lambda\right\rangle \leq \lambda_i \quad \text{for every } i \in \mathcal{I}\,. \quad (5.2)$$

Here, b and B are the optional gradient bounds introduced in Assumption 5.2.

We now define the correspondence

$$\mathcal{G}(\lambda) \triangleq \left\{ \left(\lambda_i + \langle \psi, E_i - C_i^\lambda \rangle\right)_{i \in \mathcal{I}} \;\middle|\; \psi \in \mathcal{S}(\lambda) \right\} \quad (\lambda \in \Lambda)\,. \tag{5.3}$$

In Proposition 5.11 below, we show that indeed $\mathcal{G}$ satisfies the conditions required for Kakutani's theorem. Hence, $\mathcal{G}$ has a fixed point $\lambda^* \in \Lambda$. Let $C^* \triangleq C^{\lambda^*}$ and note that, by definition of $\mathcal{G}$, there is $\psi^* \in \mathcal{S}(\lambda^*)$ such that $\langle \psi^*, E_i \rangle = \langle \psi^*, C_i^* \rangle$ for every $i \in \mathcal{I}$. As ψ^* supports the efficient allocation C^*, this gives us existence of a quasi–equilibrium in the auxiliary economy where the agents' consumption space is given by the order ideal $\tilde{\mathcal{X}}_+ \subset \mathcal{X}_+$.

Now, recall from Lemma 5.9 that, as the density ψ^* induces a price functional supporting the allocation $(C_i^*)_{i \in \mathcal{I}}$, it must take the form

$$\psi^* = \max_i \{k_i^* \nabla V_i(C_i^*)\} \tag{5.4}$$

for some constants $k_i^* \geq 0$. Note that, via the right side of (5.4), ψ^* allows a canonical extension to a nonnegative, bounded and optional process on the whole time interval $[0, \hat{T}]$. This process induces, thus, a price functional $\langle \psi^*, . \rangle$ on the 'large' consumption space $\mathcal{X}_+$.

Let us next show that, in conjunction with $(C_i^*)_{i \in \mathcal{I}}$, this functional $\langle \psi^*, . \rangle$ defines a true Arrow–Debreu equilibrium for the 'large' economy where consumption spaces are given by $\mathcal{X}_+$. To this end, fix $i \in \mathcal{I}$ and consider a plan $C_i \in \mathcal{X}_+$ which is strictly preferred to C_i^*, i.e., assume $V_i(C_i) > V_i(C_i^*)$. We have to show that $\langle \psi^*, C_i \rangle > \langle \psi^*, C_i^* \rangle = \langle \psi^*, E_i \rangle$. In fact, we have

$$0 > V_i(C_i^*) - V_i(C_i) \geq \langle \nabla V_i(C_i^*), C_i^* - C_i \rangle \geq \langle \psi^*, C_i^* - C_i \rangle / k_i^*\,,$$

where the last estimate is due to $k_i^* \nabla V_i(C_i^*) \leq \psi^*$ and to the flat–off condition in Lemma 5.9. Hence, any plan which is strictly preferred to C_i^* violates the investor's budget constraint. □

It remains to establish

Proposition 5.11 *The correspondence $\mathcal{G}$ defined by* (5.3) *satisfies the conditions of Kakutani's fixed point theorem:*

(i) For every $\lambda \in \Lambda$, $\mathcal{G}(\lambda)$ is a non–empty convex subset of Λ.

(ii) $\mathcal{G}$ is lower hemi–continuous, i.e., the graph

$$\{(\lambda, g) \mid \lambda \in \Lambda,\ g \in \mathcal{G}(\lambda)\}$$

is closed in $\Lambda \times \Lambda$.

PROOF : We adopt the notation from the preceding proof.

1. Let us focus on assertion (i) and note first that $\mathcal{G}(\lambda)$ is nonempty for every $\lambda \in \Lambda$. Indeed, we know from Lemma 5.9 that ψ_λ supports the allocation C^λ. Clearly, this property is inherited by every positive multiple of ψ_λ. Moreover, we have

$$\frac{b}{|\mathcal{I}|} \leq \psi_\lambda \leq B$$

and, thus, $\psi \triangleq \psi_\lambda / \langle B, E\rangle$ obviously satisfies the first constraint in (5.2). It also satisfies the second constraint, since by Lemma 5.4 (iii)

$$\left\langle \psi_\lambda, C_i^\lambda \right\rangle = \lambda_i \left\langle \nabla V_i(C_i^\lambda), C_i^\lambda \right\rangle \leq \lambda_i \langle B, E\rangle \ .$$

This shows that $\mathcal{S}(\lambda)$ and, hence, also $\mathcal{G}(\lambda)$ is nonempty.

Convexity of $\mathcal{G}(\lambda)$ follows from convexity of $\mathcal{S}(\lambda)$. Moreover, any $g \in \mathcal{G}(\lambda)$ satisfies $g_i \geq 0$ for every $i \in \mathcal{I}$ because of the second constraint in (5.2). In addition, $\sum_i g_i = 1$ by Lemma 5.4 (i). Hence, we have in fact $\mathcal{G}(\lambda) \subset \Lambda$ which completes the proof of assertion (i).

2. To prove assertion (ii), let $\lambda^n \in \Lambda$ and $g^n \in \mathcal{G}(\lambda^n)$ $(n = 1, 2, \ldots)$ converge to λ^0 and g^0, respectively. We have to show that $g^0 \in \mathcal{G}(\lambda^0)$. Put $C^n \triangleq C^{\lambda^n}$ and let $\psi_n \in \mathcal{S}(\lambda^n)$ be such that

$$g_i^n = \lambda_i^n + \langle \psi_n, E_i - C_i^n\rangle \quad (i \in \mathcal{I})\,. \tag{5.5}$$

Due to condition (5.2), the sequence $(\psi_n,\ n = 1, 2, \ldots)$ is dominated by the $\mathbb{P}\otimes dE$–integrable process $B/\langle B, E\rangle$. In particular, it is uniformly integrable and, by the Dunford–Pettis Theorem, there is a subsequence, again denoted by (ψ_n), which converges weakly to some ψ in $L^1(\mathbb{P} \otimes dE)$.

We shall show that ψ belongs to $\mathcal{S}(\lambda^0)$ and satisfies

$$g_i^0 = \lambda_i^0 + \left\langle \psi, E_i - C_i^0 \right\rangle \quad \text{for every } i \in \mathcal{I}. \tag{5.6}$$

Of course, this will yield assertion (ii). Our argument is based on the following result

$$\lim_n \langle \psi_n, C_i^n\rangle = \left\langle \psi, C_i^0 \right\rangle \quad \text{for every } i \in \mathcal{I} \tag{5.7}$$

which goes back to Bewley (1969) and which will be established in Step 3 of this proof.

As a first application of Bewley's claim, we note that (5.6) holds true. Indeed, granted (5.7), this follows immediately by letting $n \uparrow +\infty$ in (5.5).

Similarly, we show that ψ satisfies the second condition in (5.2) for $\lambda = \lambda^0$. Indeed as $\psi_n \in \mathcal{S}(\lambda^n)$ by definition, we know that each ψ_n satisfies $\langle \psi_n, C_i^n \rangle \le \lambda_i^n$ for every $i \in \mathcal{I}$. Given relation (5.7), we may pass to the limit $n \uparrow +\infty$ to obtain the desired inequalities

$$\left\langle \psi, C_i^0 \right\rangle \le \lambda_i^0 \quad (i \in \mathcal{I}) .$$

The first condition in (5.2) is stable with respect to weak convergence in $L^1(\mathbb{P} \otimes dE)$ and is thus inherited by ψ from ψ_n $(n = 1, 2, \ldots)$.

Concerning the supporting property of ψ, note first that, from $\psi \ge b/(|\mathcal{I}| \langle B, E \rangle)$ it follows that

$$\langle \psi, E \rangle \ge \frac{\langle b, E \rangle}{|\mathcal{I}| \langle B, E \rangle} > 0 .$$

Hence, in order to verify $\psi \in \mathcal{S}(\lambda^0)$, it only remains to show that, under the price density ψ, every allocation $C \in \tilde{\mathcal{X}}_+^{\mathcal{I}}$ which is preferred to C^0 must have a higher aggregate price than C^0. By continuity and monotonicity of preferences, it suffices to consider a *strictly* preferred allocation C in the sense that $V_i(C_i) > V_i(C_i^0)$ for every $i \in \mathcal{I}$. Due to Corollary 5.6, such an allocation C is also strictly preferred to C^n when n is large enough. As $\psi_n \in \mathcal{S}(\lambda^n)$ by assumption, each C^n is supported by ψ_n $(n = 1, 2, \ldots)$. We thus obtain that, for n sufficiently large,

$$\left\langle \psi_n, \sum\nolimits_i C_i \right\rangle \ge \langle \psi_n, E \rangle .$$

Due to the weak $L^1(\mathbb{P} \otimes dE)$–convergence $\psi_n \to \psi$, we may let $n \uparrow +\infty$ in the preceding inequality to deduce

$$\left\langle \psi, \sum\nolimits_i C_i \right\rangle \ge \langle \psi, E \rangle .$$

This shows that ψ indeed supports C^0 and, therefore, completes the proof of assertion (ii).

3. We still have to prove Bewley's claim (5.7). We follow his argument and note first that the claim is already implied by the seemingly weaker assertion

$$\limsup_n \langle \psi_n, C_i^n \rangle \le \left\langle \psi, C_i^0 \right\rangle \quad \text{for every } i \in \mathcal{I} . \tag{5.8}$$

Indeed, the aggregation property of allocations and claim (5.8) imply

$$\langle \psi, E \rangle \ge \limsup_n \sum\nolimits_i \langle \psi_n, C_i^n \rangle = \limsup_n \langle \psi_n, E \rangle . \tag{5.9}$$

Due to the weak convergence $\psi_n \to \psi$, the last term is again equal to $\langle \psi, E \rangle$. Hence, we must have equality everywhere in (5.9), and claim (5.7) follows.

In order to establish (5.8), fix $i \in \mathcal{I}$ and set $C_i^\varepsilon \triangleq C_i^0 + \varepsilon E$ for $\varepsilon > 0$. By monotonicity of preferences, we have $V_i(C_i^\varepsilon) > V_i(C_i^0)$. Due to Corollary 5.6, we also have

$$V_i(C_i^\varepsilon) > V_i(C_i^n) \quad \text{for large } n.$$

Since ψ_n supports the allocation C^n, it follows that for such n

$$\left\langle \psi_n, C_i^\varepsilon + \sum\nolimits_{j \neq i} C_j^n \right\rangle \geq \langle \psi_n, E \rangle$$

or, equivalently,

$$\langle \psi_n, C_i^\varepsilon \rangle \geq \langle \psi_n, C_i^n \rangle \ .$$

Let n tend to infinity and use the weak convergence $\psi_n \to \psi$ to conclude

$$\langle \psi, C_i^\varepsilon \rangle \geq \limsup_n \langle \psi_n, C_i^n \rangle \ .$$

Now, claim (5.8) follows from letting $\varepsilon \to 0$ in the preceding estimate.

□

5.4 Structure of Equilibrium Prices

Having established existence of an equilibrium $((C_i^*)_{i \in \mathcal{I}}, \psi^*)$, it is natural to ask, whether the induced equilibrium price functional $\langle \psi^*, . \rangle$ is continuous on our consumption space $(\mathcal{X}_+, d_{\mathcal{X}_+})$. Indeed, this is desirable from an economic point of view, since 'similar' consumption plans should have 'similar' prices. We show in this section that under two additional assumptions, we indeed have this kind of continuity of the price functional.

Our first condition is

Assumption 5.3 *The filtration $\mathbb{F} = (\mathcal{F}_t,\ 0 \leq t \leq \hat{T})$ is quasi left–continuous.*

This is an assumption on the way new information is revealed to the agents. It is satisfied, e.g., if the filtration $\mathbb{F}$ is generated by a Brownian motion or a Poisson process. Economically, an information flow corresponds to a quasi left–continuous filtration if 'information surprises' (in the sense of Hindy and Huang (1992)) occur only at times which cannot be predicted. An earthquake in New York (rather than San Francisco) is an example. The announcement of a policy change of the Federal Reserve is an example for an information surprise which occurs at a time known in advance.

The second assumption allows us to use stochastic calculus:

Assumption 5.4 *For every $C \in \mathcal{X}_+$, each utility gradient $\nabla V_i(C)$ $(i \in \mathcal{I})$ is a bounded $(\mathbb{P}, \mathbb{F})$–semimartingale with a continuous compensator of bounded variation.*

This assumption is satisfied, e.g., if all agents have Hindy–Huang–Kreps preferences with strictly positive initial level of satisfaction $\eta_i > 0$.

Theorem 5.2 *Under Assumptions 5.3 and 5.4, every equilibrium price functional*

$$C \mapsto \langle \psi^*, C \rangle \quad (C \in \mathcal{X}_+)$$

is continuous on $(\mathcal{X}_+, d_{\mathcal{X}_+})$.

PROOF : For ease of notation, we put $\psi \triangleq \psi^*$.

1. Being an equilibrium price process, ψ clearly supports its associated equilibrium allocation $C^* \triangleq (C_i^{\lambda^*})_{i \in \mathcal{I}}$. By Lemma 5.4, it thus takes the form

   $$\psi = \max_i \{k_i \nabla V_i(C_i^*)\}$$

   for suitable constants $k_i \geq 0$ $(i \in \mathcal{I})$.

   Due to Assumption 5.4, each process $\phi_i \triangleq k_i \nabla V_i(C_i^*)$ $(i \in \mathcal{I})$ allows a Doob–Meyer decomposition $\phi_i = M_i + A_i$ into a local martingale M_i and a continuous compensator A_i of bounded variation.

   Moreover, defined as the pointwise maximum of the bounded semimartingales ϕ_i, the process ψ is a bounded semimartingale, too. Hence, it can be decomposed in the form $\psi = M + A$ where M is a local martingale and A is a predictable RCLL–process of bounded variation.

   In particular, there is a localizing sequence of stopping times T_m $(m = 1, 2, \ldots)$ with $T_m(\omega) = \hat{T}$ eventually for $\mathbb{P}$–a.e. $\omega \in \Omega$ such that each of the stopped processes M^{T_m} and $M_i^{T_m}$ $(i \in \mathcal{I})$ is a (uniformly integrable) martingale on $[0, \hat{T}]$.

2. We claim that the process A in the Doob–Meyer decomposition of ψ almost surely has continuous paths. In order to prove this, it suffices to show that almost surely $A(S) = A(S-)$ for every *predictable* stopping time $S \leq \hat{T}$ because both processes $(A(t),\ (0 \leq t \leq \hat{T})$ and $(A(t-),\ 0 \leq t \leq \hat{T})$ are predictable; cf., e.g., Rogers and Williams (1987), Lemma VI.19.2.

   Now, recall that, granted quasi–left continuity of the underlying filtration, every uniformly integrable martingale almost surely does not jump at predictable times; see, e.g., Theorem VI.18.1 in Rogers and Williams (1987).

   We apply this observation first to the martingales $M_i^{T_m}$ and obtain that $\Delta M_i(S) = 0$ on $\{S \leq T_m\}$ for every $i \in \mathcal{I}$. Since, in addition, each A_i is continuous, this yields $\Delta \phi_i(S) = 0$ on $\{S \leq T_m\}$ for every $i \in \mathcal{I}$. For $m \uparrow +\infty$, this entails $\Delta \phi_i(S) = 0$ $(i \in \mathcal{I})$ and, consequently, also $\Delta \psi(S) = 0$ almost surely.

Applying the above observation to the stopped process M^{T_m} shows that also $\Delta M(S) = 0$ on $\{S \le T_m\}$. Letting $m \uparrow +\infty$ we obtain $\Delta M(S) = 0$ almost surely. Together with $\Delta\psi(S) = 0$, this implies $\Delta A(S) = 0$ and we are done.

3. We are now in a position to prove the asserted continuity of the price functional $C \mapsto \langle \psi^*, C\rangle$. To this end, let C^n $(n = 1, 2, \dots)$ converge to C^0 in $(\mathcal{X}_+, d_{\mathcal{X}_+})$. Thus, we have $L^1(\mathbb{P})$–convergence of $C^n(\hat{T})$ to $C^0(\hat{T})$ and weak*–convergence in probability of the measures dC^n to dC^0. By the usual subsequence argument, we may assume without loss of generality that both convergences hold true even almost surely.

   Note that the local martingale M is locally bounded since it is the difference of the bounded process ψ and the continuous process A. Thus, we may assume that our localizing sequence (T_m) is such that each M^{T_m} $(m = 1, 2, \dots)$ is a bounded martingale.

   For every $m = 1, 2, \dots$ we have

$$\begin{aligned}
\left|\langle \psi, C^n\rangle - \langle \psi, C^0\rangle\right| &\le \left|\mathbb{E}\int_0^{\hat{T}} \{\psi - \psi^{T_m}\}\,(dC^n - dC^0)\right| + \left|\mathbb{E}\int_0^{\hat{T}} \psi^{T_m}\,(dC^n - dC^0)\right| \\
&\le \mathbb{E}\int_{T_m}^{\hat{T}} |\psi - \psi(T_m)|\,(dC^0 + dC^n) \\
&\quad + \left|\mathbb{E}\int_0^{\hat{T}} M^{T_m}\,(dC^n - dC^0)\right| + \left|\mathbb{E}\int_0^{\hat{T}} A^{T_m}\,(dC^n - dC^0)\right| .
\end{aligned}$$

   Let us denote the preceding three summands by I, II, and III, respectively. For the first summand, we have

$$I \le \|\psi\|_\infty \mathbb{E}\left[\left(C^n(\hat{T}) + C^0(\hat{T})\right) 1_{\{T_m < \hat{T}\}}\right] .$$

   As $C^n(\hat{T}) \to C^0(\hat{T})$ in $L^1(\mathbb{P})$ by assumption, dominated convergence yields

$$\limsup_n I \le 2\|\psi\|_\infty \mathbb{E}\left[C^0(\hat{T}) 1_{\{T_m < \hat{T}\}}\right] .$$

   Using the martingale property of M^{T_m}, we may rewrite the second summand in the form

$$II = \left|\mathbb{E}\left[M(T_m)(C^n(\hat{T}) - C^0(\hat{T}))\right]\right| .$$

   Thus,

$$\limsup_n II \le \|M(T_m)\|_\infty \limsup_n \mathbb{E}|C^n(\hat{T}) - C^0(\hat{T})| = 0\,.$$

Finally, note that, due to the continuity of A, we have

$$\int_0^{\hat{T}} A^{T_m}\, dC^n \to \int_0^{\hat{T}} A^{T_m}\, dC^0 \quad (n \uparrow +\infty) \tag{5.10}$$

almost surely. Moreover, we have

$$\left| \int_0^{\hat{T}} A^{T_m}\, dC^n \right| \leq \left\| \sup_{0 \leq t \leq \hat{T}} |A^{T_m}(t)| \right\|_\infty C^n(\hat{T})\,.$$

Once again we use the $L^1(\mathbb{P})$–convergence $C^n(T) \to C^0(\hat{T})$ and deduce that the right side of this estimate defines a uniformly integrable family of random variables parameterized by n. Hence, by Lebesgue's theorem, we obtain that the convergence (5.10) holds true also in $L^1(\mathbb{P})$. For the third summand, this gives us

$$\limsup_n \mathit{III} = 0\,.$$

Summing up, we find that, for every $m = 1, 2, \ldots$,

$$\begin{aligned} \limsup_n |\langle \psi, C^n \rangle - \left\langle \psi, C^0 \right\rangle| &\leq \limsup_n I + \limsup_n \mathit{II} + \limsup_n \mathit{III} \\ &\leq 2\|\psi\|_\infty \mathbb{E}\left[C^0(\hat{T}) 1_{\{T_m < \hat{T}\}} \right] + 0 + 0\,. \end{aligned} \tag{5.11}$$

Letting $m \uparrow +\infty$ in (5.11), we get by dominated convergence that indeed

$$\limsup_n |\langle \psi, C^n \rangle - \left\langle \psi, C^0 \right\rangle| \leq 2\|\psi\|_\infty \mathbb{E}\left[C^0(\hat{T}) 1_{\bigcap_m \{T_m < \hat{T}\}} \right] = 0$$

because $T_m(\omega) = \hat{T}$ eventually for $\mathbb{P}$–a.e. $\omega \in \Omega$ by construction.

□

Properties of Equilibrium Price Processes Let us now discuss the properties of equilibrium price processes in more detail.

First, being the weighted maximum of utility gradients, the equilibrium price process is a semimartingale if gradients are semimartingales. This is an important property because it provides an equilibrium foundation for the application of stochastic calculus in mathematical finance.

Going a step further, we see from the preceding proof that the predictable compensator of the equilibrium price process is continuous if the compensator of every gradient is continuous. A fundamental question is whether this bounded variation part of equilibrium prices is even absolutely continuous because then an interest rate exists.

The gradients of Hindy–Huang–Kreps utility functionals have such a nice representation. Hence, in a one consumer world, the Hindy–Huang–Kreps approach guarantees the existence of an interest rate. Hindy and Huang (1992) even suggest that also with heterogeneous agents, the equilibrium price process would have this nice property. However, this need not always be the case. From the Tanaka formula, it follows that the maximum of semimartingales whose finite variation part is absolutely continuous can be decomposed into a local martingale, an absolutely continuous part of bounded variation and a part which depends on local time of the gradients. In general, local times are not absolutely continuous. Hence, the equilibrium price process possibly does not belong to the dual suggested by Hindy and Huang (1992). Moreover, our characterization of supporting price functionals (Lemma 5.9) shows that, in general, there may be no equilibrium whose price process is contained in the Hindy–Huang dual.

Local time arises in the decomposition of the equilibrium price process whenever the identity of the agent whose gradient determines the price changes. A very similar phenomenon has already been remarked in the time–additive setting by Karatzas, Lehoczky, and Shreve (1991). For finance theory, it implies that the money market account contains a singular component. The detailed consequences remain to be studied in future work.

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Index of Notation

$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\overline{\mathbb{R}}$	set of real numbers including $-\infty$, $+\infty$
$\triangleq$	definition
$\inf \varnothing \triangleq +\infty$	
$\sup \varnothing \triangleq -\infty$	
$\int_s^t \dots \triangleq \int_{[s,t]} \dots$	
$a \wedge b \triangleq \min\{a, b\}$	
$a \vee b \triangleq \max\{a, b\}$	
$\hat{T}$	time–horizon
$\mathbb{F} = (\mathcal{F}_t)$	filtration
$\mathcal{S}$	set of stopping times almost surely $\leq \hat{T}$
$\hat{\mathcal{S}}$	set of stopping times almost surely $< \hat{T}$
$\mathcal{S}(S)$	set of stopping times taking values in $[S, \hat{T}]$
$\mathcal{S}^{>}(S)$	set of stopping times taking values in $(S, \hat{T}]$ on $\{S < \hat{T}\}$
X	real–valued stochastic process
${}^oX$	optional projection of X
x	real–valued function on $[0, \hat{T}]$
$\breve{x}^s$	inhomogeneously convex envelope of $x\|_{[s,\hat{T}]}$
$\mathcal{M}_+$	set of deterministic consumption patterns
$d_{\mathcal{M}_+}$	Prohov–distance on $\mathcal{M}_+$
$\mathcal{C}$	set of stochastic consumption patterns
$d_{\mathcal{C}}$	metric of weak convergence in probability on $\mathcal{C}$
$\mathcal{X}_+$	set of integrable stochastic consumption patterns
$d_{\mathcal{X}_+}$	metric of weak convergence in probability plus L^1–convergence of total masses on $\mathcal{X}_+$
$\tilde{\mathcal{X}}_+$	order ideal associated with $\mathcal{X}_+$
$\preceq$	ordering on $\mathcal{M}_+$ and $\mathcal{C}$
$(.,.)$, $\langle .,. \rangle$	bracket operators on $\mathcal{M}_+$ and $\mathcal{C}$

U	utility functional
u	felicity function
∇U	subgradient of U
V	expected utility functional
∇V	optional subgradient of V
$\mathrm{Dom}(V)$	domain of functional V
$\mathcal{A}(w)$	budget–feasible consumption patterns
$\mathcal{Z}$	set of feasible allocations
$\mathcal{I}$	finite set of economic agents
Λ	set of normalized weights
$\mathbb{P}$	objective probability
$\mathbb{E}$	expectation with respect to $\mathbb{P}$
$\mathcal{P}$	some set of $\mathbb{P}$–equivalent probability measures
$\mathbb{P}^*$, $\hat{\mathbb{P}}$	elements of $\mathcal{P}$
$\mathbb{E}^*$, $\hat{\mathbb{E}}$	expectation with respect to $\mathbb{P}^*$, $\hat{\mathbb{P}}$, respectively
Ψ	price–functional
ψ	state–price density
$\pi(.)$, $\pi^*(.)$	Laplace–exponents of X under $\mathbb{P}$ and $\mathbb{P}^*$, respectively
C	consumption pattern or allocation
E	endowment stream
C^λ	efficient allocation for agents' weights λ
C^L	consumption plan tracking level process L
$Y(C)(t)$	level of satisfaction derived from consumption plan C up to time t
$Y^L \triangleq Y(C^L)$	
η	initial level of satisfaction
β	rate of satisfaction decay
$B(t) \triangleq \int_0^t \beta(s)\,ds$	
δ	rate of time–preference
α	parameter for risk–aversion
r	interest rate
θ	market price of risk